Linear Elasticity III - Conformal mapping, Beam theory

This TA completes the 4 week series on linear elasticity. We believe that the problems we'll be solving today, together with the thermo-elastic problem of the previous TA, really capture a lot of the tools used to tackle real problems, and also provide an insight into the beauty and complexity of even such a simple linear theory. Next week we'll be moving forward to non-linear elasticity, so don't get too comfortable.

1 Complex representation of scalar elasticity

We study a case of scalar elasticity, where $\boldsymbol{u} = u_z(x,y)\boldsymbol{e}_z$. The strains are

$$\varepsilon_{yz} = \frac{1}{2}(\partial_y u_z + \partial_z u_y) = \frac{1}{2}\partial_y u_z \ , \tag{1}$$

$$\varepsilon_{xz} = \frac{1}{2}(\partial_x u_z + \partial_z u_x) = \frac{1}{2}\partial_x u_z \ , \tag{2}$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0 \ . \tag{3}$$

We have seen in class that $\nabla^2 u_z = 0$, that is, u_z is a harmonic function. This means that we can write u_z as

$$u_z = 2\Re(\phi) = \phi(z) + \overline{\phi(z)}, \qquad z = x + iy \ , \tag{4}$$

where ϕ is an analytic complex function. We will use the Cauchy-Riemman equations, that tell us that

$$\partial_x \phi = -i\partial_y \phi = \phi' \ , \tag{5}$$

$$\partial_x \overline{\phi} = \overline{\partial_x \phi} = i\partial_y \overline{\phi} = \overline{\phi}' \ , \tag{6}$$

and therefore the stresses are

$$\begin{aligned}
\sigma_{xz} &= \mu\partial_x u_z = \mu\left(\partial_x\phi + \partial_x\overline{\phi}\right) = \mu\left(\phi' + \overline{\phi}'\right) = 2\mu\Re(\phi') \ , \\
\sigma_{yz} &= \mu\partial_y u_z = \mu\left(\partial_y\phi + \partial_y\overline{\phi}\right) = i\mu\left(\phi' - \overline{\phi}'\right) = -2\mu\Im(\phi') \ , \\
\Rightarrow &\qquad 2\mu\phi' = \sigma_{xz} - i\sigma_{yz} \ ,
\end{aligned} \tag{7}$$

and all other components vanish.

If our domain contains a free boundary, given by a curve that is parameterized by $x(s), y(s)$ with s being arc-length parametrization, then the normal to the boundary is given by $\boldsymbol{n} = (\partial_s y, -\partial_s x)$. On the boundary we thus have

$$\begin{aligned}
0 &= \sigma_{zx}n_x + \sigma_{zy}n_y \\
&= \mu\left[\left(\partial_x\phi + \partial_x\overline{\phi}\right)\partial_s y - \left(\partial_y\phi + \partial_y\overline{\phi}\right)\partial_s x\right] \\
&= \mu\left[\left(-i\partial_y\phi + i\partial_y\overline{\phi}\right)\partial_s y - \left(i\partial_x\phi - i\partial_x\overline{\phi}\right)\partial_s x\right] \\
&= -i\mu\left[\left(\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial s}\right) - \left(\frac{\partial\overline{\phi}}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial\overline{\phi}}{\partial x}\frac{\partial x}{\partial s}\right)\right] \\
&= -\mu\left(\frac{\partial\phi}{\partial s} - \frac{\partial\overline{\phi}}{\partial s}\right) = 2\mu\frac{\partial\Im(\phi)}{\partial s} \ ,
\end{aligned} \tag{8}$$

so on the boundary $\Im(\phi)$ is constant. Since ϕ is only given up to an additive constant, we can choose $\Im(\phi) = 0$, or, in other words, $\phi = \overline{\phi}$ on the boundary. We see that solving for the displacement field is equivalent to finding an analytic function whose imaginary part is constant on the boundary.

2 Conformal mapping: Inglis crack

(Refernce: Marder & Fineberg, Physics Reports 1999)

Complex treatment of 2D elasticity is very useful because Laplace's equation is conformally invariant, so one can use conformal mappings to deform the region over which we need to solve the equation into a more convenient geometry. Here we'll see an application of this method, which is called the Inglis (mode III) problem. In 1913 Charles Inglis solved the general problem of an elliptic hole in an infinite plate subject to distant loading. His solution turned out to be one of the cornerstones of fracture mechanics, and was later used and generalized by the works of Griffith, Irwin, and others.

So let's look at an infinite plane with an elliptic hole, subject to antiplane shear $\sigma_{yz} = \sigma_\infty$ at $y \to \pm\infty$. As working with ellipses is unpleasant, we want to find a conformal mapping that maps the region outside the ellipse to a region outside a circle. Luckily, such a mapping is well known, and is given by

$$z = f(\omega) = R\left(\omega + \frac{\rho}{\omega}\right) \ , \tag{9}$$

$$\omega = f^{-1}(z) = \frac{z}{2R} + \sqrt{\left(\frac{z}{2R}\right)^2 - \rho} \ . \tag{10}$$

f maps the unit circle in the ω-plane to an ellipse with axes $R(1 \pm \rho)$ in the z-plane. $0 \le \rho \le 1$ is a parameter that measures the ellipse's eccentricity[1] - when $\rho = 0$ the ellipse is a circle, while for $\rho = 1$ it is a 1D crack of length $4R$. The conformal mapping is shown in Fig. (1).

The crux of the conformal mapping technique is that while in the real coordinates the geometry is elliptic (and thus complicated), in the ω-plane the domain is a circle (simple!), and therefore we want to reformulate the problem in terms of ω. That is, we want to describe ϕ as a function of ω, by the mapping $\phi(\omega) = \phi(\omega(z))$.

On the hole's boundary, which is the unit circle in ω-plane, we have

$$\phi(\omega) = \overline{\phi(\omega)} = \overline{\phi}(\overline{\omega}) = \overline{\phi}(1/\omega) \ , \tag{11}$$

because on the unit circle $\overline{\omega} = 1/\omega$. The property (11) can be analytically extended to all the ω-plane.

What are the singularities of ϕ? Outside the hole, it must be completely regular, except at infinity where it diverges as $\phi \sim z$. This is because Eq. (7) tells us that far from the hole we have $\partial_z \phi \propto \sigma/\mu$, and therefore we conclude that

$$\phi \approx -i\frac{\sigma_\infty}{\mu} z \approx -i\frac{\sigma_\infty}{\mu} R\omega, \qquad \text{for } \omega, z \to \infty \ . \tag{12}$$

[1] Note that ρ isn't the eccentricity as usually defined in geometry, which is $e = \frac{2\sqrt{\rho}}{\rho+1}$.

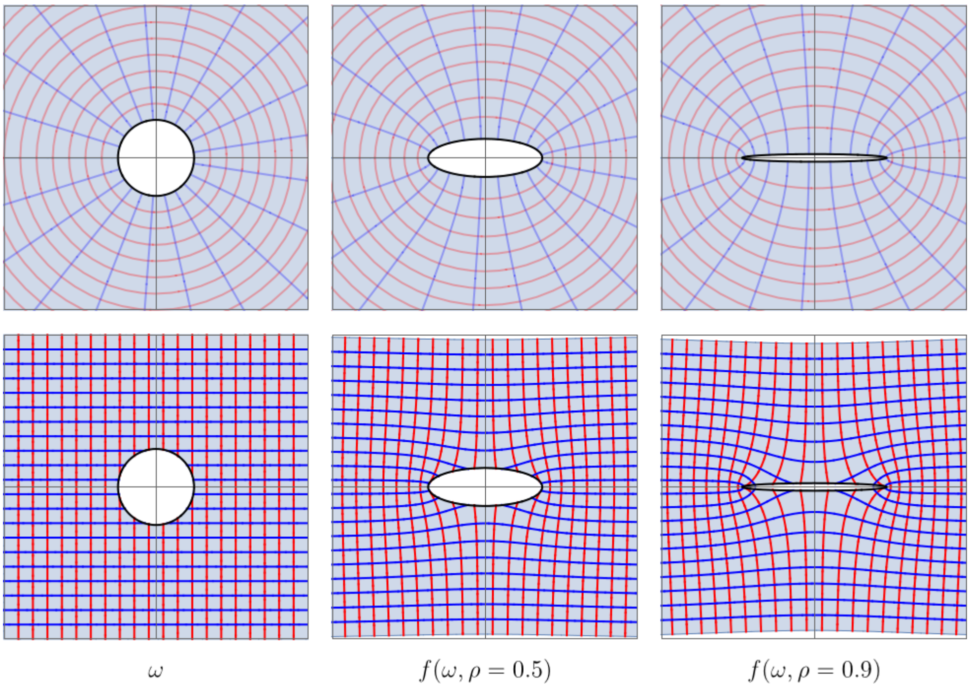

Figure 1: The conformal mapping. Polar lines (top row) and the Cartesian lines (bottom row) are shown. Note that, after the mapping, the lines remain perpendicular.

Using the analytical continuation of Eq. (11), we get that

$$\overline{\phi}(1/\omega) \approx -i\frac{\sigma_\infty}{\mu}R\omega, \qquad \text{for } \omega \to \infty \ , \tag{13}$$

or equivalently,

$$\phi(\omega) \approx i\frac{\sigma_\infty R}{\mu\omega}, \qquad \text{for } \omega \to 0 \ , \tag{14}$$

and there are no other singularities inside the unit circle. Having determined all the possible singularities of ϕ, it is determined up to an additive constant. It must be

$$\phi(\omega) = i\frac{\sigma_\infty R}{\mu}\left(\frac{1}{\omega} - \omega\right) \ . \tag{15}$$

As discussed above, another way of finding ϕ is to find a function whose imaginary part vanishes on the boundary on the hole, i.e. on the unit circle. The function $i\left(1/\omega - \omega\right)$ fits this requirement, therefore, it is exactly the function we're looking for, up to a multiplicative factor which we have obtained from the external BC.

We can now calculate the displacement field in the "real" coordinate z by joining Eqs. (15) and (10):

$$u_z = 2\Re\left\{-i\frac{\sigma_\infty R}{\mu}\left(\zeta + \sqrt{\zeta^2 - \rho} - \frac{1}{\overline{\zeta + \sqrt{\zeta^2 - \rho}}}\right)\right\} \ , \qquad \text{where } \zeta \equiv \tfrac{z}{2R} \ . \tag{16}$$

What is the stress at the tip of the ellipse? We can differentiate $u_z(z)$ of Eq. (16) explicitly, but this gives a nasty expression that is very difficult to understand. It is simpler to use the conformal property of the mapping:

$$\begin{aligned}
\partial_z\phi(z) &= \partial_z\phi(\omega(z)) = \phi'(\omega)\frac{\partial\omega}{\partial z} \ , \\
\phi'(\omega) &= -i\frac{\sigma_\infty R}{\mu}\left(1+\frac{1}{\omega^2}\right) \ , \\
\frac{\partial\omega}{\partial z} &= \left(\frac{\partial z}{\partial w}\right)^{-1} = \frac{1}{f'(\omega)} \ , \\
f'(\omega) &= R(1-\frac{\rho}{\omega^2}) \ , \\
\phi'(z) &= \frac{-i\frac{\sigma_\infty R}{\mu}\left(1+\frac{1}{\omega^2}\right)}{R(1-\frac{\rho}{\omega^2})} = -\frac{i\sigma_\infty}{\mu}\,\frac{\omega(z)^2+1}{\omega(z)^2-\rho} \ .
\end{aligned} \tag{17}$$

Note that in the last equation ω is a function of z.

Now let's examine the solution. One thing we would like to know is where in space is the stress maximal. Clearly, ϕ' diverges for $w=\pm\sqrt{\rho}$, but remember that $\rho<1$ and our domain is outside the unit circle, so this point is inside the hole. Some trivial algebra shows that the ϕ' is maximal for $\omega=\pm1$, which are, not surprisingly, the closest points outside the unit circle to $\pm\rho$. When $\omega=\pm1$ we have $z=\pm R(1+\rho)$ - these are the horizontal tips of the ellipse. The stresses there are

$$\begin{aligned}
\sigma_{xz}-i\sigma_{yz} = \mu\phi' = -\sigma_\infty\frac{2i}{1-\rho} \Rightarrow \\
\sigma_{xz}=0 \qquad , \qquad \sigma_{yz}=\frac{2\sigma_\infty}{1-\rho} \ .
\end{aligned} \tag{18}$$

The case $\rho=0$ gives $\sigma_{yz}=2\sigma_\infty$, in accordance with what was done in class. In the opposite extremity, $\rho\to1$, the stress field diverges (but the displacement doesn't). We see that the stress at the tip decreases with the radius there. An interesting consequence of this is that in order to arrest a crack from propagating, one can drill a hole at its tip (!). This will reduce the radius of curvature at the tip and weaken the singularity.

The limiting case $\rho\to1$ is of particular interest, as it describes a 1-dimensional cut in the material. It is known in the literature as Mode III crack. The power with which σ_{zy} diverges in the case $\rho=1$ can be easily obtained. In this case we have

$$\phi = -\frac{iR\sigma_\infty}{\mu}\sqrt{\frac{z^2}{R^2}-4} \ . \tag{19}$$

Plugging in $z=2R\,(1+\delta)$ (where $\delta\in\mathbb{C}$) and keeping the leading order in δ gives

$$\begin{aligned}
\phi = -i\frac{2\sqrt{2}R\sigma_\infty}{\mu}\sqrt{\delta}+O\left(\delta^{3/2}\right) \Rightarrow \\
\sigma_{yz} \sim \frac{\sigma_\infty}{\sqrt{2}\sqrt{\delta}} \ .
\end{aligned} \tag{20}$$

The fact that near the crack tip the stress field diverges as the square root of the distance from the crack tip, and that the displacement field is regular, is of general applicability, and is true for static cracks in all loading configurations. The square-root divergence is a consequence of the branch-cut at the crack surface.

3 Euler-Bernoulli Beam Theory

We saw earlier in the course how we can reduce the theory of elasticity to 2D. This significantly simplifies the equations and makes everything more tractable. If we have a 1D system, we can do even better. The theory of 1D elastic systems is elegantly dealt with by the Euler-Bernoulli beam theory. It was a long standing problem that was attacked by Galileo and Leonardo, and finally solved by Euler and Bernoulli around 1750. It is considered a corner-stone in civil engineering (and was used in the structural design of the Eiffel tower!).

The general purpose is to describe the deformation of a beam by a single curve $\gamma(s)$ in 3D ("elastic line"), and write an effective balance equation that takes care of all the deformation details by using the curve's curvature, torsion, and so on. There are two ways to go about it. The first is to use symmetry considerations to write down a 1D constitutive law. The problem is that we are left with unknown parameters, but the physics is clear. The second option is to "integrate out" the details of the cross-section deformation.

3.1 Short derivation

Here's a way to derive the equation in a short way, following our philosophy of continuum physics. For simplicity we will assume that the deformation is limited to a plane. This is not necessary, but it greatly simplifies everything. Consider then a beam under a small deflection. Choose x to be the direction of the rod's axis, and let it deform in the xz-plane, as shown in Fig. 2.

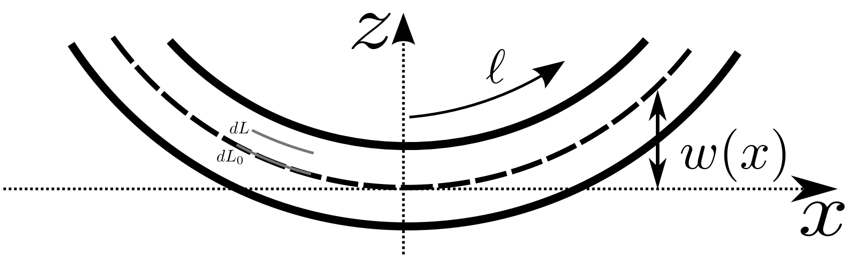

Figure 2: Description of a beam's plane deflection.

The kinetic energy of the rod is given by

$$E_k = \int \frac{1}{2}\lambda\,(\dot{w})^2\,dx \ , \tag{21}$$

where λ is the linear mass density. The potential energy is a bit more subtle. It clearly cannot depend on w explicitly but only on its derivatives, because it should be invariant under $w \to w + const$. How about w'? For a free rod, still no, because we can rotate a flat rod to have a non zero slope. On the other hand, if the rod is under tension T, the potential energy density is $\frac{1}{2}Tw'(x)^2$, just as is the case for a stretched string[2]. Is this term enough? Yes, if the resistance arises only from the tension, like in a string. If

[2] As this is shown in undergrad level courses I won't repeat it here, but you can find the derivation here.

we want to consider a solid rod, which has some stiffness resisting bending, we need to consider the next derivative, w''. We know that the energy should be invariant under $w \to -w$, so we conclude that the energy density, at least to lower order, should be $\frac{1}{2}\kappa(w'')^2$, where κ is some (possibly space-dependent) material parameter of dimensions energy $\times$ length. As we expect this rigidity to be proportional to the elastic modulus of the material, we may write $\kappa = EI$, where E is the Young's modulus and I is a parameter of dimensions length^4. If there are external forces there's an extra term to the potential energy $f^{ext}w$, so the potential energy is

$$E_p \approx \left[\int \frac{1}{2}Tw'(x)^2 + \frac{1}{2}EI(x)\left(w''\right)^2 - f^{ext}w\right] dx \ . \tag{22}$$

We can then write the Lagrangian

$$\mathcal{L} = \int (E_k - E_p)dx \ , \tag{23}$$

and apply the Euler-Lagrange equations[3] to it. The result is

$$\lambda\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + T\frac{\partial^2 w}{\partial x^2} + f_z^{ext} \ . \tag{24}$$

Notice that we have no a priori way of knowing what $I(x)$ will be, but the 1D physics can be understood without knowing it. A systematic integration of the cross section will give us the value of I, as will be shown in Subsec. 3.3.

3.2 Examples

The Euler-Bernoulli theory makes seemingly complicated problems very easy to solve. You might not appreciate that yet because you didn't yet solve many problems in elasticity, but I will try to impress you by showing quite a few examples in quite a short time.

3.2.1 Buckling

Buckling is the phenomenon when a beam that is subject to axial compression fails via large lateral deflection (see Fig 3). Obviously, it has enormous implications for structural engineering. Euler-Bernoulli equations give a good prediction for the onset of instability.

In this scenario the deformation is dominated by a constant compressive force $P = -T$[4], and external forces vanish in the bulk. That is, we have $f^{ext} = 0$, $\ddot{w} = 0$, and Eq. (54) reads

$$EIw'''' + Pw'' = \frac{\partial^2}{\partial x^2}\left(EIw'' + Pw\right) = 0 \ . \tag{25}$$

[3] Notice that we have a second derivative in the Lagrangian, so the E-L equations are a bit different than the usual form. You can see details here.

[4] I changed the notation from T to P because now the force is compressive, and that means $T < 0$. Clearly, this is just for notational convenience, and has no real meaning.

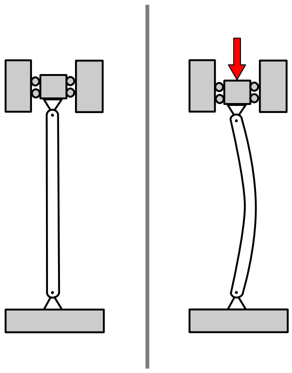

Figure 3: Buckling

We still need to specify the boundary conditions at the ends of the rod. For example, let us take pinned boundary conditions, $w = w'' = 0$ (described in Fig 3). Note that Eq. (25) always admits the solution $w = 0$, which we already know - this is simple uniaxaial compression. However, another possible solution to this equation is

$$w(x) = A \sin\left(x\sqrt{\frac{P}{EI}}\right) \ , \tag{26}$$

which exists only if half the wavelength the wavelength $\pi\sqrt{EI/P}$ divides L. That is, if P is smaller than $P_c = \pi^2 EI/L^2$, the beam will not buckle and will deform by simple uniaxial compression. When P surpasses this value, the uniaxial solution becomes unstable and the beam will buckle with a finite amplitude. This is called Euler buckling, and is a textbook engineering estimate for strength of columns. Experiments show that this is a good estimate when the aspect ratio of the column is very large (> 10). The value of P_c depends on the boundary conditions at the ends of the rod - you can do the exercise for clamped edges yourselves.

Note also an interesting point: when P is negative, i.e. T is positive, the beam is under tension rather than compression and buckled solutions do not exist. This is of course very intuitive - you know that sticks do not buckle when you stretch them! - but it's nice to see how it comes out so cleanly from the algebra.

The loss of stability can also be seen by performing a formal stability analysis, as discussed with Eran. Consider the Dynamical equation

$$\lambda \ddot{w} + EIw'''' + Pw'' = 0 \ , \tag{27}$$

and consider the solution $e^{ikx+\Lambda t}$. As the beam is finite, k takes on discrete values[5] $k = \pi n/L$, leading to the characteristic equation

$$\pi^4 EIn^4 + \lambda L^4 \Lambda^2 - \pi^2 L^2 n^2 P = 0 \Rightarrow \Lambda = \pm\frac{\pi\,|n|\,\sqrt{L^2 P - \pi^2 EIn^2}}{\sqrt{\lambda} L^2} \ . \tag{28}$$

In general, a condition for stability is that for all values of k, $\Re\left[\Lambda\right] \leq 0$. In our case, Λ is either purely real or purely imaginary. This, combined with the fact that we have

[5] We are assuming the physical solution is the imaginary part, so it agrees with the BC.

positive and negative solutions, means that we are stable if for all values on n the term in the $\sqrt{}$ is negative, so

$$L^2 P - \pi^2 E I n^2 < 0 \quad \forall n \in \mathbb{Z} \Rightarrow P < \frac{\pi^2 E I}{L^2} \ , \tag{29}$$

just as discussed above.

3.2.2 Static deformation of beams

Let's calculate the static deformation of a horizontal beam of length L that is clamped at its edges and is subject to gravity (panel a in Fig. (4)).

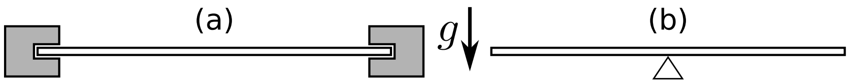

Figure 4: Two setups of beam deformation under gravity.

Just imagine the pain in the neck it would be to fully solve the 3D problem! Anyhow, in this case we have

$$\boldsymbol{f}^{ext}(x) = -\lambda g \hat{\boldsymbol{z}} \ , \tag{30}$$

and $\ddot{w} = 0, \ T = 0$. Eq. (54) thus reads

$$EI \frac{\partial^4 w(x)}{\partial x^4} = -\lambda g \ , \qquad w\left(\pm \frac{L}{2}\right) = w'\left(\pm \frac{L}{2}\right) = 0 \ . \tag{31}$$

The solution of (31) is clearly a 4th-order polynomial in x, with the leading term $-\frac{1}{4!}\frac{\lambda g}{EI} x^4$. We also know that this polynomial has roots of order 2 at $x = \pm L/2$. This leaves no choice but the solution

$$w(x) = -\frac{1}{4!}\frac{\lambda g}{EI}\left[x^2 - \left(\frac{L}{2}\right)^2\right]^2 \ . \tag{32}$$

In a similar way, let's try to calculate the profile of a beam that is supported at its middle (panel b in Fig. 4). This corresponds to the equation

$$EIw'''' = -\lambda g + Lg\lambda\delta(x) \ , \tag{33}$$

where δ term represents the point force exerted by the support. Eqs. (47) and (49) tell us that w'' is proportional to the internal torque and w''' to the force in the beam. At the endpoints both of these vanish, so the boundary conditions are

$$w''\left(\pm \frac{L}{2}\right) = w'''\left(\pm \frac{L}{2}\right) = 0 \ . \tag{34}$$

One can then trivially integrate Eq. (33) and plug in the boundary conditions to solve for the coefficients of the resulting polynomial. The algebra is completely trivial but nonetheless annoying. One can also exploit the symmetry of the problem and solve the problem only for one half of the beam, where the equation is

$$EIw'''' = -\lambda g \ , \qquad\qquad w'(0) = w''(L/2) = w'''(L/2) = 0 \ . \tag{35}$$

The solution, up to an additive constant, must be

$$w = -\frac{\lambda g}{4!EI} x^2(x^2 + Ax + B) \ . \tag{36}$$

The constants A, B are easily obtained from the boundary conditions at $x = L/2$, the result being

$$w = -\frac{g\lambda}{24EI} x^2 \left(x^2 - 2Lx + \frac{3}{2}L^2 \right) \ , \tag{37}$$

and the deflection at $x = L/2$ equals $\Delta z = -\frac{\lambda g}{128EI}L^4$. Did you expect the deflection to scale as L^4? Note that we have an odd term in what is supposed to be an even function. How do you explain this?

3.2.3 Waves

The dynamic Euler Bernoulli equation is basically a wave equation:

$$\lambda \frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2} \right) + T\frac{\partial^2 w}{\partial x^2} \ . \tag{38}$$

Note that a standard wave equation for a stretched wire is recovered if we forget about the EIw'''' term, that is, if we care only about tension and neglect the bending rigidity. This is exactly the physical assumption that you made when you modeled a stretched wire in your first-year physics course!

If EI is constant, we can guess a solution of the form $w = Ae^{ikx+i\omega t}$, and get the dispersion relation

$$\omega^2 = \frac{EI}{\lambda}k^4 + \frac{T}{\lambda}k^2 \ . \tag{39}$$

You see that for large k's bending is dominant, and for small k's tension takes over.

3.3 Detailed derivation

Here I try to distill a semi-heuristic derivation of the equations, which is a bit simplistic, but all the physics is in there. For a detailed derivation, see Landau& Lifschitz #7 §18, or Timoshenko & Godier chap. 12.

3.3.1 Force Balance

Let's look at a small portion of the rod, as shown in Fig. 2. We will allow only small deformations, i.e. small w and $\partial_x w$. Let's denote the total force on a cross-section by $\boldsymbol{F}(\ell)$ and the total torque by $\boldsymbol{M}(\ell)$. They are given by

$$F_i = \int \sigma_{ix} dS \ , \tag{40}$$

$$M_i = \int \mathcal{E}_{ijk}\sigma_{jx}(\vec{r} - \vec{r}_0)_k dS \ , \tag{41}$$

where $\vec{r}_0$ is the cross section's center of mass, $\mathcal{E}$ is the Levi-Civita tensor, and dS is a surface element *perpendicular* to ℓ. Force balance on a beam element clearly implies

$$\frac{\partial \boldsymbol{F}}{\partial \ell} + \boldsymbol{f}^{ext} = 0 \; , \tag{42}$$

where $\boldsymbol{f}^{ext}$ is the external force density on the rod. If the external forces do not have an axial component, Eq. (42) reduces to $\boldsymbol{F} = const$. The total torque on an element of length $d\ell$ is given by

$$\boldsymbol{M}(x + d\ell) - \boldsymbol{M}(x) - \boldsymbol{F}(x + d\ell) \times d\vec{\ell} \; .$$

Moment balance thus reads

$$\frac{\partial \boldsymbol{M}}{\partial \ell} = \boldsymbol{F} \times \vec{t} \; , \tag{43}$$

where $\vec{t}$ is a unit tangent vector which is equal, to first order, to $\vec{t} \approx (1, 0, w')$. Differentiating Eq. (43) gives

$$\frac{\partial^2 \boldsymbol{M}}{\partial \ell^2} = \frac{\partial \boldsymbol{F}}{\partial \ell} \times \vec{t} + \boldsymbol{F} \times \frac{\partial \vec{t}}{\partial \ell} \; . \tag{44}$$

3.3.2 Relation between force and deformation

We have derived the basic macroscopic force-balance equation. Now, the goal is to relate $\boldsymbol{F}$ and $\boldsymbol{M}$ to the details of the beam's deformation. The length of a line element dL_0 in the axial direction, which is situated at height z relative to the mid-line (see Fig. 2), is changed by the deformation to be $dL = \frac{R-z}{R} dL_0$, where R is the local radius of curvature. We therefore have

$$\varepsilon_{xx} \approx \frac{dL - dL_0}{dL_0} = -\frac{z}{R} \; , \tag{45}$$

$$\sigma_{xx} \approx -E\frac{z}{R} \; . \tag{46}$$

If the rod is almost parallel to the x axis, and the deformation of the mid-line is given by $z = w(x)$, then we can approximate to first order in $\partial_x w$ that $R^{-1} \approx \partial_{xx} w$ and $\partial_l \approx \partial_x$. The torque is therefore

$$M_y(x) = \int -z\sigma_{xx} dS = -\int E\frac{z^2}{R} dS = -\frac{EI}{R} \approx -EI\frac{\partial^2 w}{\partial x^2} \; , \tag{47}$$

with the definition

$$I = \int z^2 dS \; , \tag{48}$$

I is called *the second moment of area.* I depends on the geometry of the cross section in a similar way to the moment of inertia. If the deformation is not restricted to a single plane, I should be treated as a tensorial quantity, in much the same way the moment of inertia shows its tensorial character when rotation is not confined to a plane. We will not bother ourselves with these complications.

Also, the product EI is sometimes called bending rigidity, or bending stiffness. It has dimensions of energy $\times$ length. If the cross-section of the body is not uniform, I may be x dependent.

The vertical force applied by one element on its neighbor is obtained by combining Eqs. (43) and (47):

$$F_z(x) = \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) \ . \tag{49}$$

3.3.3 Closing up

We now have all the necessary components. Say the beam is under axial tensile force T (the stress is T/A where A is the cross-section area). The F is then given by

$$\vec{F} = \left(-T, 0, \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\right) \ . \tag{50}$$

We assume also that f^{ext} is in the z direction, i.e. has no axial component and is limited to the xz plane that is under consideration. The tangent $\hat{t}$ is given by

$$\hat{t} = (1, 0, w') \ , \tag{51}$$

the norm of which is 1 to linear order. Plugging the last two equations into (44) we get

$$\begin{aligned} \boldsymbol{M}'' &= \left(0, 0, -f^{ext}\right) \times (1, 0, w') + \left(-T, 0, \frac{\partial}{\partial x}\left(EIw''\right)\right) \times (0, 0, w'') \\ &= \left(0, f^{ext} + Tw'', 0\right) \ . \end{aligned} \tag{52}$$

In addition, we know that $\boldsymbol{M}$ has only a y component, and the left-hand-side of (44) reads

$$\boldsymbol{M}'' = -\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) \ . \tag{53}$$

We therefore conclude that

$$-\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + T\frac{\partial^2 w}{\partial x^2} + f_z^{ext} = 0 \ . \tag{54}$$

Usually this equation is written with $T = 0$ and is then called the *Euler-Bernoulli beam equation*.

Physically, we can interpret it (when $T = 0$) as $\partial_x F_z = f_z^{ext}$. That is, the external forces balance out exactly the forces caused by internal deformation. If we don't assume static equilibrium, we can add inertia:

$$\lambda\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + T\frac{\partial^2 w}{\partial x^2} + f_z^{ext} \ , \tag{55}$$

where λ is the linear mass density. This is identical to Eq. (24). The only difference is that now we know I from a more basic theory.

Note that this theory does not take into account the angular momentum balance caused by rotation. This correction is included in an extension of the theory called the Timoshenko Beam theory.

3.4 A short note about the analogy with splines

If you have a collection of data and you want to draw a line through all the points ("a guide to the eye..."), but you don't have a clue about what to do between data points, what do you do? If you're a good scientist then you simply don't draw anything. But if you're not (and the first sign of that is that you're using Excel), then you'll ask Excel to draw *something* and it will draw a spline.

What is a spline? A spline on a data set is a piecewise 3rd order polynomial that passes through all the data points, is continuous, and has a continuous derivative. It needs to be 3rd order because you need four parameters for each segment (between data points) - two values and two derivatives.

Note that if you were to constrain an elastic beam to pass through your data points, this is exactly what it would do!