Non-Equilibrium Continuum Physics

Extended lecture notes by Eran Bouchbinder

(Dated: April 5, 2025)

This course is intended to introduce graduate students to the essentials of modern continuum physics, with a focus on non-equilibrium phenomena in solids and within a thermodynamic perspective. Special focus is given to emergent phenomena, where collective many-body systems reveal physical principles that cannot be inferred from the microscopic physics of a small number of degrees of freedom. General concepts and principles — such as conservation laws, symmetries, material frame-indifference, dissipation inequalities and non-equilibrium behaviors, spatiotemporal symmetry-breaking instabilities and configurational forces — are emphasized. Examples cover a wide range of physical phenomena and applications in diverse disciplines. The power of field theory as a mathematical structure that does not make direct reference to microscopic length scales well below those of the phenomenon of interest is highlighted. Some basic mathematical tools and techniques are introduced. The course highlights essential ideas and basic physical intuition. Together with courses on fluid mechanics and soft condensed matter, a broad background and understanding of continuum physics will be established.

The course will be given within a framework of 12-13 two-hour lectures and 12-13 two-hour tutorial sessions with a focus on problem-solving. No prior knowledge of the subject is assumed. Basic knowledge of statistical thermodynamics, vector calculus, partial differential equations, dynamical systems and complex analysis is required.

These extended lecture notes (book draft) are self-contained and in principle no other materials are needed.

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Tutorials: Wednesday, 09:15–11:00

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Final grade:

About 6-7 problems sets throughout the semester (40%) and final exam/assignment (60%).

Attendance:

Expected and highly encouraged (both lectures and tutorial sessions).

General Principles and Concepts

I. INTRODUCTION: BACKGROUND AND MOTIVATION

We start by considering the course's title. By 'non-equilibrium' we refer to physical phenomena that cannot be properly treated in the framework of equilibrium thermodynamics. That is, we refer to phenomena that involve irreversible processes and dissipation. We will, however, make an effort to adhere as much as possible to thermodynamic formulations (i.e,we will not focus on purely dynamical systems) and also devote time to reversible phenomena (both because they are often missing from current physics education and because they set the stage for discussing irreversible phenomena). By 'continuum' we refer to the scientific approach that treats macroscopic phenomena without making explicit reference to the discreteness of matter or more generally to microscopic length and time scales. This also implies that we focus on collective phenomena that involve spatially extended systems and a macroscopic number of degrees of freedom (atoms, molecules, grains etc.). We therefore treat materials as continua and use the language of field theory to describe the phenomena of interest. A crucial concept in this context is that of emergent phenomena, which refers to the fundamental idea that collective many-body systems reveal laws/behavior that cannot be inferred from microscopic laws of physics and a small number of degrees of freedom; that is, "More is Different", adopting the famous title of Philip W. Anderson (see Science **177**, 393 (1972)).

'Physics' is surely a bit too broad here, yet it represents the idea that the tools and concepts that will be discussed have a very broad range of applications in different branches of physics. In addition, the topics considered can be discussed from various perspectives — such as applied mathematics, engineering sciences and materials science —, but we will adopt a physicist perspective. To make 'physics' even more specific in the present context, we note that we will mainly focus on thermal and mechanical phenomena, rather than electrical, magnetic or chemical phenomena. By 'thermal' and 'mechanical' — or 'thermomechanical' we refer to material phenomena that involve deformation, material and heat flow and failure, and where the driving forces are thermal and mechanical in nature. 'Classical continuum mechanics' typically refers to 'solid mechanics' and 'fluid mechanics' from a classical (i.e,non-quantum) physics perspective. In this course we will mainly focus on solids in the broadest sense of the word.

The word 'solid' is not easily defined. The most intricate aspect of such a definition is that it involves an observation timescale (at least if we do not consider single crystals). However, for the purpose of this course, it will be sufficient to define a solid as a material that can support shear forces over sufficiently long timescales. We therefore do not focus on Newtonian fluids and very soft materials (though we certainly mention them), both of which are discussed in complementary courses. Nevertheless, we will discuss solid phenomena such as visco-elasticity and nonlinear elasticity.

Why should one study the subjects taught in this course? Well, there are many (good) reasons. Let us mention a few of them. First, macroscopic physics deals with emergent phenomena that cannot be understood from microscopic laws applied to a small number of constituent elements (degrees of freedom). That is, macroscopic systems feature new qualitative coarse-grained properties and dynamics. This is a deep conceptual, to some extent even philosophical, issue that should be systematically introduced. Second, many of the macroscopic phenomena around us are both non-equilibrium and thermomechanical in nature. This course offers tools to understand some of these phenomena. Third, continuum physics phenomena, and solid-related phenomena in particular, are ubiquitous in many branches of science and therefore understanding them may be very useful for researchers in a broad range of disciplines. Fourth, the conceptual and mathematical tools of non-equilibrium thermodynamics and field theory are extremely useful in many branches of science, and thus constitute an important part of scientific education. Finally, some of the issues discussed in this course are related to several outstanding unsolved problems. Hence, the course will expose students to the beauty and depth of a fundamental and active field of research. It would be impossible to even scratch the surface of the huge ongoing solid-related activity. Let us mention a few examples: **(i)** It has been quite recently recognized that the mechanics of living matter, cells in particular, plays a central role in biology. For example, it has been discovered that the stiffness of the substrate on which stem cells grow can significantly affect their differentiation. **(ii)** Biomimetics: researchers have realized that natural/biological systems exhibit superior mechanical properties, and hence aim at mimicking the design principles of these systems in man-made ones. For example, people have managed to build superior adhesives based on Gecko's motion on a wall. People have succeeded in synthesizing better composite materials based on the structures observed in hard tissues, such as cortical bone and dentin. **(iii)** The efforts to understand the physics of driven disordered systems (granular materials, molecular glasses, colloidal suspensions etc.) are deeply related to one of the most outstanding questions in non-equilibrium statistical physics. **(iv)** People have recently realized there are intimate relations between geometry and mechanics. For example, by controlling the intrinsic metric of materials, macroscopic shapes can be explained and designed. **(v)** The rupture of materials and interfaces has a growing influence on our understanding and control of the world around us. For example, there are exciting developments in understanding Earthquakes, the failure of interfaces between two tectonic plates in the Earth's crust **(vi)** Developments in understanding the plastic deformation of amorphous and crystalline solids offer deep new insights about strongly nonlinear and dissipative systems, and open the way to new and exciting applications.

Unfortunately, due to time limitations, the course cannot follow a *historical perspective* which highlights the evolution of the developed ideas. These may provide very important scientific, sociological and psychological insights, especially for research students and young researchers. Whenever possible, historical notes will be made.

II. MATHEMATICAL PRELIMINARIES: TENSOR ANALYSIS

The fundamental assumption of continuum physics is that under a wide range of conditions we can treat materials as *continuous* in space and time, disregarding their discrete structure and time-evolution at microscopic length and time scales, respectively. Therefore, we can ascribe to each point in space-time physical properties that can be described by continuous functions, i.e,*fields*. This implies that derivatives are well defined and hence that we can use the powerful tools of differential calculus. In order to understand what kind of continuous functions, hereafter termed fields, should be used, we should bear in mind that physical laws must be independent of the position and orientation of an observer, and the time of observation (note that we restrict ourselves to classical physics, excluding the theory of relativity). We are concerned here, however, with the mathematical objects that allow us to formulate this and related principles. Most generally, we are interested in the language that naturally allows a mathematical formulation of continuum physical laws. The basic ingredients in this language are *tensor fields*, which are the major focus on the opening part of the course.

Tensor fields are characterized, among other things, by their *order* (sometimes also termed *rank*). Zero-order tensors are *scalars*, for example the temperature field $T(\boldsymbol{x}, t)$ within a body, where $\boldsymbol{x}$ is a 3-dimensional Euclidean space and t is time. First-order tensors are *vectors*, for example the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ of a fluid. **Why do we need to consider objects that are higher-order than vectors?** The best way to answer this question is through an example. Consider a material areal element and the force acting on it (if the material areal element is a surface element, then the force is applied externally and if the material areal element is inside the bulk material, then the force is exerted by neighboring material). The point is that both the areal element and the force acting on it are basically vectors, i.e,they both have an orientation (the orientation of the areal element is usually quantified by the direction of the normal to it). Therefore, in order to characterize this physical situation one should say that a force in the ith direction is acting on a material areal element whose normal points in the jth direction. The resulting object is defined using two vectors, but it is not a vector itself. We need a higher-order tensor to describe it.

Our main interest here is second-order tensors, which play a major role in continuum physics. A second-order tensor $\boldsymbol{A}$ can be viewed as a linear operator or a linear function that maps a vector, say $\boldsymbol{u}$, to a vector, say $\boldsymbol{v}$,

$$\boldsymbol{v} = \boldsymbol{A}\boldsymbol{u} \ . \tag{2.1}$$

Linearity implies that

$$\boldsymbol{A}(\alpha\boldsymbol{u}+\boldsymbol{v}) = \alpha\boldsymbol{A}\boldsymbol{u}+\boldsymbol{A}\boldsymbol{v}\ , \tag{2.2}$$

for every scalar α and vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. For brevity, second-order tensors will be usually referred to simply as tensors (zero-order tensors will be termed scalars, first-order tensors will be termed vectors and higher than second-order tensors will be explicitly referred to according to their order).

The most natural way to define (or express) tensors in terms of vectors is through the *dyadic* (or *tensor*) product of orthonormal base vectors $\{\boldsymbol{e}_i\}$

$$\boldsymbol{A} = A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\ , \tag{2.3}$$

where Einstein summation convention is adopted, $\{A_{ij}\}$ is a set of numbers and $\{i,j\}$ run over space dimensions. For those who feel more comfortable with Dirac's Bra-Ket notation, the dyadic product above can be also written as $\boldsymbol{A} = A_{ij}\,|\boldsymbol{e}_i\!><\!\boldsymbol{e}_j|$. In general, the dyad $\boldsymbol{u}\otimes\boldsymbol{v}$ is defined as

$$\boldsymbol{u}\otimes\boldsymbol{v} = \boldsymbol{u}\boldsymbol{v}^T\ , \tag{2.4}$$

where vectors are assumed to be represented by column vectors and the superscript T denotes the transpose operation. If $\{\boldsymbol{e}_i\}$ is an orthonormal set of Cartesian base vectors, we have (for example)

$$\boldsymbol{e}_2\otimes\boldsymbol{e}_3 = \boldsymbol{e}_2\boldsymbol{e}_3^T = \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0&0&1\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\ . \tag{2.5}$$

Therefore, second-order tensors can be directly *represented* by matrices. Thus, tensor algebra essentially reduces to matrix algebra. It is useful to note that for every three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ we have

$$\boldsymbol{u}\otimes\boldsymbol{v}\,\boldsymbol{w} = (\boldsymbol{v}\cdot\boldsymbol{w})\,\boldsymbol{u}\ . \tag{2.6}$$

where $\cdot$ is the usual inner (dot) product of vectors. In the Bra-Ket notation the above simply reads $|\boldsymbol{u}\!><\!\boldsymbol{v}|\boldsymbol{w}\!>$. This immediately allows us to rewrite Eq. (2.1) as

$$v_i\boldsymbol{e}_i = \boldsymbol{v} = \boldsymbol{A}\boldsymbol{u} = (A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j)(u_k\boldsymbol{e}_k) = A_{ij}u_k(\boldsymbol{e}_j\cdot\boldsymbol{e}_k)\,\boldsymbol{e}_i = A_{ij}u_j\boldsymbol{e}_i\ , \tag{2.7}$$

which shows that the matrix representation preserves known properties of matrix algebra ($v_i = A_{ij}u_j$). The matrix representation allows us to define additional tensorial operators. For example, we can define

$$\mathrm{tr}(\boldsymbol{A}) \equiv \boldsymbol{e}_k \cdot (A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\, \boldsymbol{e}_k = A_{ij} <\boldsymbol{e}_k|\boldsymbol{e}_i><\boldsymbol{e}_j|\boldsymbol{e}_k> = A_{ij}\delta_{ik}\delta_{jk} = A_{kk} \ , \tag{2.8}$$

$$\boldsymbol{A}^T = (A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j)^T = A_{ij}\boldsymbol{e}_j \otimes \boldsymbol{e}_i = A_{ji}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \ , \tag{2.9}$$

$$\boldsymbol{AB} = (A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j)(B_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l) = A_{ij}B_{kl}\delta_{jk}\boldsymbol{e}_i \otimes \boldsymbol{e}_l = A_{ij}B_{jl}\boldsymbol{e}_i \otimes \boldsymbol{e}_l \ . \tag{2.10}$$

We can define the *double dot product* (or the *contraction*) of two tensors as

$$\begin{aligned} \boldsymbol{A} : \boldsymbol{B} &= (A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (B_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l) \equiv A_{ij}B_{kl}(\boldsymbol{e}_i \cdot \boldsymbol{e}_k)(\boldsymbol{e}_j \cdot \boldsymbol{e}_l) \\ &= A_{ij}B_{kl}\delta_{ik}\delta_{jl} = A_{ij}B_{ij} = \mathrm{tr}(\boldsymbol{A}\,\boldsymbol{B}^T) \ . \end{aligned} \tag{2.11}$$

This is a natural way of generating a scalar out of two tensors, which is the tensorial generalization of the usual vectorial dot product (hence the name). It plays an important role in the thermodynamics of deforming bodies. Furthermore, it allows us to project a tensor on a base dyad

$$(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : \boldsymbol{A} = (\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (A_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l) = A_{kl}(\boldsymbol{e}_i \cdot \boldsymbol{e}_k)(\boldsymbol{e}_j \cdot \boldsymbol{e}_l) = A_{kl}\delta_{ik}\delta_{jl} = A_{ij} \ , \tag{2.12}$$

i.e,to extract a component of a tensor.

We can now define the identity tensor as

$$\boldsymbol{I} = \delta_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \ , \tag{2.13}$$

which immediately allows to define the inverse of a tensor (when it exists) following

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} \ . \tag{2.14}$$

The existence of the inverse is guaranteed when $\det \boldsymbol{A} \neq 0$, where the determinant of a tensor is defined using the determinant of its matrix representation. Note also that one can decompose any second-order tensor to a sum of symmetric and skew-symmetric (antisymmetric) parts as

$$\boldsymbol{A} = \boldsymbol{A}_{sym} + \boldsymbol{A}_{skew} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T) + \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T) \ . \tag{2.15}$$

Occasionally, physical constraints render the tensors of interest symmetric, i.e,$\boldsymbol{A} = \boldsymbol{A}^T$. In this case, we can diagonalize the tensor by formulating the eigenvalue problem

$$\boldsymbol{A}\boldsymbol{a}_i = \lambda_i \boldsymbol{a}_i \ , \tag{2.16}$$

where $\{\lambda_i\}$ and $\{\boldsymbol{a}_i\}$ are the eigenvalues (principal values) and the orthonormal eigenvectors (principal directions), respectively. This problem is analogous to finding the roots of

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = -\lambda^3 + \lambda^2 I_1(\boldsymbol{A}) - \lambda I_2(\boldsymbol{A}) + I_3(\boldsymbol{A}) = 0 \ , \tag{2.17}$$

where the *principal invariants* $\{I_i(\boldsymbol{A})\}$ are given by

$$I_1(\boldsymbol{A}) = \mathrm{tr}(\boldsymbol{A}), \quad I_2(\boldsymbol{A}) = \frac{1}{2}\left[\mathrm{tr}^2(\boldsymbol{A}) - \mathrm{tr}(\boldsymbol{A}^2)\right], \quad I_3(\boldsymbol{A}) = \det(\boldsymbol{A}) \ . \tag{2.18}$$

Note that the symmetry of $\boldsymbol{A}$ ensures that the eigenvalues are real and that an orthonormal set of eigenvectors can be constructed. Therefore, we can represent any symmetric tensor as

$$\boldsymbol{A} = \lambda_i \, \boldsymbol{a}_i \otimes \boldsymbol{a}_i \ , \tag{2.19}$$

assuming no degeneracy. This is called the *spectral decomposition* of a symmetric tensor $\boldsymbol{A}$. It is very useful because it represents a tensor by 3 real numbers and 3 unit vectors. It also allows us to define functions of tensors. For example, for positive definite tensors ($\lambda_i > 0$), we can define

$$\ln(\boldsymbol{A}) = \ln(\lambda_i) \, \boldsymbol{a}_i \otimes \boldsymbol{a}_i \ , \tag{2.20}$$

$$\sqrt{\boldsymbol{A}} = \sqrt{\lambda_i} \, \boldsymbol{a}_i \otimes \boldsymbol{a}_i \ . \tag{2.21}$$

In general, one can define functions of tensors that are themselves scalars, vectors or tensors. Consider, for example, a scalar function of a tensor $f(\boldsymbol{A})$ (e.g., the energy density of a deforming solid). Consequently, we need to consider *tensor calculus*. For example, the derivative of $f(\boldsymbol{A})$ with respect to $\boldsymbol{A}$ is a tensor which takes the form

$$\frac{\partial f}{\partial \boldsymbol{A}} = \frac{\partial f}{\partial A_{ij}} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \ . \tag{2.22}$$

The differential of $f(\boldsymbol{A})$ is a scalar and reads

$$df = \frac{\partial f}{\partial \boldsymbol{A}} : d\boldsymbol{A} = \frac{\partial f}{\partial A_{ij}} dA_{ij} \ . \tag{2.23}$$

Consider then a tensorial function of a tensor $\boldsymbol{F}(\boldsymbol{A})$, which is encountered quite regularly in continuum physics. Its derivative $\boldsymbol{D}$ is defined as

$$\begin{aligned} \boldsymbol{D} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{A}} &= \frac{\partial \boldsymbol{F}}{\partial A_{ij}} \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \frac{\partial F_{kl}}{\partial A_{ij}} \boldsymbol{e}_k \otimes \boldsymbol{e}_l \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \ , \\ \Longrightarrow D_{klij} &= \frac{\partial F_{kl}}{\partial A_{ij}} \ , \end{aligned} \tag{2.24}$$

which is a fourth-order tensor.

We will now define some differential operators that either produce tensors or act on tensors. First, consider a vector field $\boldsymbol{v}(\boldsymbol{x})$ and define its gradient as

$$\nabla \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{v}}{\partial x_j} \otimes \boldsymbol{e}_j = \frac{\partial v_i}{\partial x_j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \ , \tag{2.25}$$

which is a second-order tensor. Then, consider the divergence of a tensor

$$\nabla\cdot\boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial x_k}\boldsymbol{e}_k = \frac{\partial A_{ij}}{\partial x_k}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\boldsymbol{e}_k = \frac{\partial A_{ij}}{\partial x_j}\boldsymbol{e}_i \ , \tag{2.26}$$

which is a vector. The last two objects are extensively used in continuum physics.

The tensorial version of Gauss' theorem for relating volume integrals to surface integrals reads

$$\int_V \nabla\cdot\boldsymbol{A}\, dV = \oint_S \boldsymbol{A}\boldsymbol{n}\, dS \ , \tag{2.27}$$

where V and S are the volume and the enclosing surface, respectively, and $\boldsymbol{n}$ is the outward unit normal to the surface. Obviously, the theorem is satisfied for scalars and vectors as well. It would be useful to recall also Stokes' theorem for relating line integrals to surface integrals

$$\int_S (\nabla\times\boldsymbol{v})\cdot\boldsymbol{n} dS = \oint_l \boldsymbol{v}\cdot d\boldsymbol{l} \ , \tag{2.28}$$

where S and l are the surface and its bounding curve, respectively, and $\boldsymbol{n}$ is the outward unit normal to the surface.

Finally, we should ask ourselves how do tensors transform under a coordinate transformation (from $\boldsymbol{x}$ to $\boldsymbol{x}'$)

$$\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x} \ , \tag{2.29}$$

where $\boldsymbol{Q}$ is a proper ($\det\boldsymbol{Q}=1$) orthogonal transformation matrix $\boldsymbol{Q}^T=\boldsymbol{Q}^{-1}$ (note that it is not a tensor). In order to understand the transformation properties of the orthonormal base vectors $\{\boldsymbol{e}_i\}$ we first note that

$$\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x} \Longrightarrow \boldsymbol{x} = \boldsymbol{Q}^T\boldsymbol{x}' \Longrightarrow x_i = Q^T_{ij}x'_j = Q_{ji}x'_j \ . \tag{2.30}$$

A vector is an object that retains its (geometric) identity under a coordinate transformation. For example, a general position vector $\boldsymbol{r}$ can be *represented* using two different base vectors sets $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}'_i\}$ as

$$\boldsymbol{r} = x_i\boldsymbol{e}_i = x'_j\boldsymbol{e}'_j \ . \tag{2.31}$$

Using Eq. (2.30) we obtain

$$x_i\boldsymbol{e}_i = (Q_{ji}x'_j)\boldsymbol{e}_i = x'_j(Q_{ji}\boldsymbol{e}_i) = x'_j\boldsymbol{e}'_j \ , \tag{2.32}$$

which implies

$$\boldsymbol{e}'_i = Q_{ij}\boldsymbol{e}_j \ . \tag{2.33}$$

In order to derive the transformation law for tensors representation we first note that tensors, like vectors, are objects that retain their (geometric) identity under a coordinate transformation and therefore we must have

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = A'_{ij}\boldsymbol{e}'_i \otimes \boldsymbol{e}'_j \ . \tag{2.34}$$

Using Eq. (2.33) we obtain

$$\boldsymbol{A} = A'_{ij}\boldsymbol{e}'_i \otimes \boldsymbol{e}'_j = A'_{ij}Q_{ik}\boldsymbol{e}_k \otimes Q_{jl}\boldsymbol{e}_l = (A'_{ij}Q_{ik}Q_{jl})\boldsymbol{e}_k \otimes \boldsymbol{e}_l \ . \tag{2.35}$$

which implies

$$A_{kl} = A'_{ij}Q_{ik}Q_{jl} \ . \tag{2.36}$$

This is the transformation law for the components of a tensor and in many textbooks it serves as a definition of a tensor. Eq. (2.36) can be written in terms of matrix representation as

$$[\boldsymbol{A}] = \boldsymbol{Q}^T[\boldsymbol{A}]'\boldsymbol{Q} \Longrightarrow [\boldsymbol{A}]' = \boldsymbol{Q}[\boldsymbol{A}]\boldsymbol{Q}^T \ , \tag{2.37}$$

where $[\cdot]$ is the matrix representation of a tensor with respect to a set of base vectors. Though we did not make the explicit distinction between a tensor and its matrix representation earlier, it is important in the present context; $[\boldsymbol{A}]$ and $[\boldsymbol{A}]'$ are different representations of the same object, the tensor $\boldsymbol{A}$, but **not** different tensors. An isotropic tensor is a tensor whose representation is independent of the coordinate system, i.e.,

$$A_{ij} = A'_{ij} \quad \text{or} \quad [\boldsymbol{A}] = [\boldsymbol{A}]' \ . \tag{2.38}$$

We note in passing that in the present context we do not distinguish between covariant and contravariant tensors, a distinction that is relevant for non-Cartesian tensors (a *Cartesian tensor* is a tensor in three-dimensional Euclidean space for which a coordinate transformation $\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x}$ satisfies $\partial x'_i/\partial x_j = \partial x_j/\partial x'_i$).

III. MOTION, DEFORMATION AND STRESS

Solid materials are deformed under applied driving forces. In order to describe the deformation of solids, consider a body at a given time, typically in the absence of external driving forces, and assign to each material point a position vector $\boldsymbol{X}$ with respect to some fixed coordinate system (i.e., we already use the continuum assumption). For simplicity, set $t\!=\!0$. You can think of $\boldsymbol{X}$ as the label of each point in the body.

At $t\!>\!0$ the body experiences some external forcing that deforms it to a state in which each material point is described by a position vector $\boldsymbol{x}$. We then define the **motion** as the following mapping

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X},t) = \boldsymbol{\varphi}(\boldsymbol{X},t) \ . \tag{3.1}$$

The vector function $\boldsymbol{\varphi}(\cdot)$ maps each point in the initial state $\boldsymbol{X}$ to a point in the current state $\boldsymbol{x}$ at $t\!>\!0$. This immediately implies that $\boldsymbol{X} = \boldsymbol{\varphi}(\boldsymbol{X}, t=0)$, i.e., at time $t=0$ $\boldsymbol{\varphi}(\cdot)$ is the identity vector. The initial state $\boldsymbol{X}$ is usually termed the *reference/undeformed configuration* and the current state is termed the *current/deformed configuration*. We assume that $\boldsymbol{\varphi}(\cdot)$ is a one-to-one mapping, i.e,that it can be inverted

$$\boldsymbol{X} = \boldsymbol{\varphi}^{-1}(\boldsymbol{x},t) \ . \tag{3.2}$$

The inverse mapping $\boldsymbol{\varphi}^{-1}(\cdot)$ tells us where a material point, that is currently at $\boldsymbol{x}$, was at time $t=0$.

Obviously, our goal is to describe the properties and spatiotemporal dynamics of the current state of the material at $t>0$. This can be done either using the $\boldsymbol{X}$ labeling, which is called the *material (Lagrangian) description*, or the $\boldsymbol{x}$ positions, which is called the *spatial (Eulerian) description*. The choice between these descriptions is a matter of convenience. For a given physical phenomenon under consideration, one description may be more convenient than the other. We will discuss this issue later in the course.

A quantity of fundamental importance is the *displacement field* defined as

$$\boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{x}(\boldsymbol{X},t) - \boldsymbol{X} \ . \tag{3.3}$$

This material description can be converted into a spatial description following

$$\boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{x}(\boldsymbol{X},t) - \boldsymbol{X} = \boldsymbol{x} - \boldsymbol{X}(\boldsymbol{x},t) = \boldsymbol{U}(\boldsymbol{\varphi}^{-1}(\boldsymbol{x},t),t) = \boldsymbol{u}(\boldsymbol{x},t) \ . \tag{3.4}$$

Note that $\boldsymbol{U}$ and $\boldsymbol{u}$ are different functions of different arguments, though their values are the same. The velocity and acceleration fields are defined as

$$\boldsymbol{V}(\boldsymbol{X},t) = \partial_t \boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{v}(\boldsymbol{x},t) \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{X},t) = \partial_{tt} \boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{a}(\boldsymbol{x},t)\ . \tag{3.5}$$

The corresponding spatial descriptions can be easily obtained using $\boldsymbol{\varphi}(\cdot)$.

The material time derivative D/Dt, which we abbreviate by D_t, is defined as the partial derivative with respect to time, keeping the Lagrangian coordinate $\boldsymbol{X}$ fixed. For a material field $\mathcal{F}(\boldsymbol{X},t)$ (scalar or vector. For a tensor, see the discussion of objectivity/frame-indifference later in the course) we have

$$D_t \mathcal{F}(\boldsymbol{X},t) \equiv \left(\partial_t \mathcal{F}(\boldsymbol{X},t)\right)_{\boldsymbol{X}}\ , \tag{3.6}$$

where we stress that $\boldsymbol{X}$ is held fixed here. This derivative represents the time rate of change of a field $\mathcal{F}$, as seen by an observer moving with a particle that was at $\boldsymbol{X}$ at time $t = 0$. We can then ask ourselves what happens when we operate with the material derivative on an Eulerian field $f(\boldsymbol{x},t)$. Using the definition in Eq. (3.6), we obtain

$$\begin{aligned} \frac{Df(\boldsymbol{x},t)}{Dt} &= \left(\frac{\partial f(\boldsymbol{\varphi}(\boldsymbol{X},t),t)}{\partial t}\right)_{\boldsymbol{X}=\boldsymbol{\varphi}^{-1}(\boldsymbol{x},t)} \\ &= \left(\frac{\partial f(\boldsymbol{x},t)}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial f(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\right)_t \left(\frac{\partial \boldsymbol{\varphi}(\boldsymbol{X},t)}{\partial t}\right)_{\boldsymbol{X}=\boldsymbol{\varphi}^{-1}(\boldsymbol{x},t)}\ . \end{aligned} \tag{3.7}$$

The last term in the above expression is the velocity field, cf. Eq. (3.5), implying that

$$\frac{D(\cdots)}{Dt} = \frac{\partial(\cdots)}{\partial t} + v_k \frac{\partial(\cdots)}{\partial x_k}\ . \tag{3.8}$$

The second contribution on the right hand side of the above equation is termed the *convective rate of change* and hence the material derivative of an Eulerian field is sometimes called the *convective derivative*. Finally, note that since the material derivative of an Eulerian field is just the total time derivative of the Eulerian field, viewing $\boldsymbol{x}(t)$ as a function of time, it is sometimes denoted by a superimposed dot, i.e,$\dot{f}(\boldsymbol{x},t) = D_t f(\boldsymbol{x},t)$. If $f(\boldsymbol{x},t)$ is the velocity field we obtain

$$D_t \boldsymbol{v}(\boldsymbol{x},t) = \partial_t \boldsymbol{v}(\boldsymbol{x},t) + \boldsymbol{v}(\boldsymbol{x},t) \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v}(\boldsymbol{x},t)\ . \tag{3.9}$$

The latter nonlinearity is very important in fluid mechanics, though it appears also in the context of elasto-plasticity. Note that we distinguish between the *spatial gradient* $\nabla_{\boldsymbol{x}}$ and the *material gradient* $\nabla_{\boldsymbol{X}}$, which are different differential operators. Fluid flows are usually described using an Eulerian description. Nevertheless, Lagrangian formulations can be revealing, see for example the Lagrangian turbulence simulation at: http://www.youtube.com/watch?v=LHIIn72dRPk

In order to discuss the physics of deformation we need to know how material line elements change their length and orientation. Therefore, we define the *deformation gradient tensor* $\boldsymbol{F}$ that maps an infinitesimal line element in the reference configuration $d\boldsymbol{X}$ to an infinitesimal line element in the deformed configuration

$$d\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{X}, t)d\boldsymbol{X} \ . \tag{3.10}$$

Hence,

$$\boldsymbol{F}(\boldsymbol{X}, t) = \nabla_{\boldsymbol{X}}\varphi(\boldsymbol{X}, t) \ . \tag{3.11}$$

As will become apparent later in the course, $\boldsymbol{F}$ is not a proper tensor, but rather a two-point tensor, i.e,a tensor that relates two configurations. We can further define the *displacement gradient tensor* as

$$\boldsymbol{H}(\boldsymbol{X}, t) = \nabla_{\boldsymbol{X}}\boldsymbol{U}(\boldsymbol{X}, t) \ , \tag{3.12}$$

which implies

$$\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{H} \ . \tag{3.13}$$

Here and elsewhere $\boldsymbol{I}$ is the identity tensor. The deformation gradient tensor $\boldsymbol{F}$ describes both the rotation and the stretching of a material line element, which also implies that it is not symmetric. From a basic physics perspective, it is clear that interaction potentials are sensitive to the relative distance between particles, but not to local rigid rotations. Consequently, we are interested in separating rotations from stretching, where the latter quantifies the change in length of material elements. We can, therefore, decompose $\boldsymbol{F}$ as

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R} \ , \tag{3.14}$$

where $\boldsymbol{R}$ is a proper rotation tensor, $\det \boldsymbol{R} = +1$, and $\boldsymbol{U}$ (should not be confused with the displacement field) and $\boldsymbol{V}$ are the right and left stretch tensors, respectively (which are of course symmetric). This is the so-called *polar decomposition*. Note that

$$\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}, \quad \boldsymbol{U} = \boldsymbol{U}^T, \quad \boldsymbol{V} = \boldsymbol{V}^T, \quad \boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T \ . \tag{3.15}$$

Therefore, $\boldsymbol{U}$ and $\boldsymbol{V}$ have the same eigenvalues (principal stretches), but different eigenvectors (principal directions). Hence, we can write the spectral decomposition as

$$\boldsymbol{V} = \lambda_i \, \boldsymbol{N}_i \otimes \boldsymbol{N}_i, \tag{3.16}$$

$$\boldsymbol{U} = \lambda_i \, \boldsymbol{M}_i \otimes \boldsymbol{M}_i, \tag{3.17}$$

with

$$\lambda_i > 0, \quad \boldsymbol{N}_i \otimes \boldsymbol{N}_i = \boldsymbol{R}\boldsymbol{M}_i \otimes \boldsymbol{R}\boldsymbol{M}_i \ . \tag{3.18}$$

A. Strain measures

At this stage, we are interested in constructing quantities that are based on the stretch tensors discussed above in order to be able, eventually, to define the energy of deformation. For this aim, we need to discuss strain measures. Unlike displacements and stretches, which are directly measurable quantities (whether it always make physical sense and over which timescales, will be discussed later), *strain measures* are *concepts* that are defined as function of the stretches, and may be conveniently chosen differently in different physical situations. The basic idea is simple; we would like to come up with a measure of the relative change in length of material line elements. Consider first the scalar (one-dimensional) case. If the reference length of a material element is ℓ_0 and its deformed length is $\ell = \lambda\ell_0$, then a simple strain measure is constructed by

$$g(\lambda) = \frac{\ell - \ell_0}{\ell_0} = \lambda - 1 \ . \tag{3.19}$$

This definition follows our intuitive notion of strain, i.e,(i) It is a monotonically increasing function of the stretch λ (ii) It vanishes when $\lambda = 1$. It is, however, by no means unique. In fact, every monotonically increasing function of λ which reduces to the above definition when λ is close to unity, i.e,satisfies $g(1) = 0$ and $g'(1) = 1$, would qualify. These conditions ensure that upon linearization, all strain measures agree. For example,

$$g(\lambda) = \int_{\ell_0}^{\ell} \frac{d\ell'}{\ell'} = \ln\left(\frac{\ell}{\ell_0}\right) = \ln\lambda \ , \tag{3.20}$$

$$g(\lambda) = \frac{\ell^2 - \ell_0^2}{2\ell_0^2} = \frac{1}{2}\left(\lambda^2 - 1\right) \ . \tag{3.21}$$

Obviously, there are infinitely many more. The three possibilities we presented above, however, are well-motivated from a physical point of view. Before explaining this, we note that the scalar (one-dimensional) definitions adopted above can be easily generalized to rotationally invariant tensorial forms as

$$\boldsymbol{E}_B = (\lambda_i - 1)\, \boldsymbol{M}_i \otimes \boldsymbol{M}_i = \boldsymbol{U} - \boldsymbol{I}, \tag{3.22}$$

$$\boldsymbol{E}_H = (\ln\lambda_i)\, \boldsymbol{M}_i \otimes \boldsymbol{M}_i = \ln\boldsymbol{U}, \tag{3.23}$$

$$\boldsymbol{E} = \frac{1}{2}(\lambda_i^2 - 1)\, \boldsymbol{M}_i \otimes \boldsymbol{M}_i = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{I}\right) = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) \ . \tag{3.24}$$

$\boldsymbol{E}_B$ is the Biot (extensional) strain tensor. It is the most intuitive strain measure. Its main disadvantage is that it cannot be directly expressed in terms of the deformation gradient tensor $\boldsymbol{F}$, but rather has to be calculated from it by a polar decomposition. $\boldsymbol{E}_H$ is the Hencky (logarithmic) strain (which is also not expressible in terms of $\boldsymbol{F}$ alone). Its one-dimensional form, Eq. (3.20), clearly demonstrates that $d\boldsymbol{E}_H$ is an incremental strain that measures incremental changes in the length of material line elements relative to their *current* length. Finally, $\boldsymbol{E}$ is the Green-Lagrange (metric) strain. While it is difficult to motivate its one-dimensional form, Eq. (3.21), its tensorial form has a clear physical meaning. To see this, consider infinitesimal line elements of size $d\ell$ and $d\ell'$ in the reference and deformed configurations respectively and construct the following measure of the change in their length

$$(d\ell')^2 - (d\ell)^2 = dx_i dx_i - dX_i dX_i = F_{ij} dX_j F_{ik} dX_k - dX_j \delta_{jk} dX_k = \tag{3.25}$$
$$2dX_j \left[\frac{1}{2}\left(F_{ij}F_{ik} - \delta_{jk}\right)\right] dX_k = 2dX_j \left[\frac{1}{2}\left(F^T_{ji}F_{ik} - \delta_{jk}\right)\right] dX_k \equiv 2dX_j E_{jk} dX_k \ .$$

Therefore,

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) = \frac{1}{2}\left(\boldsymbol{C} - \boldsymbol{I}\right) = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{I}\right) = \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T + \boldsymbol{H}^T\boldsymbol{H}) \ , \tag{3.26}$$

where $\mathbf{C} \equiv \boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy-Green deformation tensor. So $\boldsymbol{E}$ is indeed a *material* metric strain tensor. Further note that $\boldsymbol{E}$ is quadratically nonlinear in the displacement gradient $\boldsymbol{H}$. The linear part of $\boldsymbol{E}$

$$\boldsymbol{\varepsilon} \equiv \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T) \tag{3.27}$$

is the linear (infinitesimal) strain tensor, which is not a true strain measure (as it is *not* rotationally invariant), but nevertheless is the basic object in the linearized field theory of elasticity (to be discussed later in the course). We can easily derive the *spatial* counterpart of $\boldsymbol{E}$, by having $(d\ell')^2 - (d\ell)^2 \equiv 2dx_j e_{jk} dx_k$, with (prove)

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right) = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{b}^{-1}\right) \ . \tag{3.28}$$

$\boldsymbol{b} \equiv \boldsymbol{F}\boldsymbol{F}^T$ is the left Cauchy-Green deformation tensor (also termed the Finger tensor, which is sometimes denoted by $\boldsymbol{B}$). $\boldsymbol{e}$, known as the Euler-Almansi strain tensor, is a *spatial* metric strain tensor.

The deformation gradient tensor $\boldsymbol{F}$ maps objects from the undeformed to the deformed configuration. For example, consider a volume element in the deformed configuration (assume $\boldsymbol{F}$ has already been diagonalized)

$$d\boldsymbol{x}^3 = dx_1 dx_2 dx_3 = F_{11} dX_1 F_{22} dX_2 F_{33} dX_3 = J(\boldsymbol{X}, t) d\boldsymbol{X}^3 \ , \tag{3.29}$$

where

$$J(\boldsymbol{X}, t) = \det \boldsymbol{F}(\boldsymbol{X}, t)\ . \tag{3.30}$$

Consider then a surface element in the undeformed configuration $d\boldsymbol{S} = dS\,\boldsymbol{N}$, where dS an infinitesimal area and $\boldsymbol{N}$ is a unit normal. The corresponding surface element in the deformed configuration is $d\boldsymbol{s} = ds\,\boldsymbol{n}$. To relate these quantities, we consider an arbitrary line element $d\boldsymbol{X}$ going through $d\boldsymbol{S}$ and express the spanned volume element by a dot product $d\boldsymbol{X}^3 = d\boldsymbol{S} \cdot d\boldsymbol{X}$. $d\boldsymbol{X}$ maps to $d\boldsymbol{x}$, which spans a corresponding volume element in the deformed configuration $d\boldsymbol{x}^3 = d\boldsymbol{s} \cdot d\boldsymbol{x}$. Using Eq. (3.29), the relation $d\boldsymbol{s} \cdot \boldsymbol{F} d\boldsymbol{X} = \boldsymbol{F}^T d\boldsymbol{s} \cdot d\boldsymbol{X}$ (i.e,$ds_i F_{ij} dX_j = F^T_{ji} ds_i dX_j$) and the fact that $d\boldsymbol{X}$ is an arbitrary line element, we obtain

$$d\boldsymbol{S} = J^{-1} \boldsymbol{F}^T d\boldsymbol{s}\ . \tag{3.31}$$

The spatial velocity gradient $\boldsymbol{L}(\boldsymbol{x}, t)$ is defined as

$$\boldsymbol{L} \equiv \frac{\partial \boldsymbol{v}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1}\ . \tag{3.32}$$

The symmetric part of $\boldsymbol{L}$, $\boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T)$, is an important quantity called the *rate of deformation* tensor. The anti-symmetric part of $\boldsymbol{L}$, $\boldsymbol{W} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T)$, is called the *spin (vorticity)* tensor.

B. The concept of stress

As was mentioned at the beginning of this section, material deformation is induced by forces. In order to describe and quantify forces at the continuum level we need the concept of *stress* (sketched earlier in section II to motivate the need for tensors). Consider a surface element $d\boldsymbol{s}$ in the deformed configuration. It is characterized by an outward normal $\boldsymbol{n}$ and a unit area ds. The surface element can be a part of the external boundary of the body or a part of an imaginary internal surface. The force acting on it, either by external agents in the former case or by neighboring material in the latter case, is denoted by $d\boldsymbol{f}$. We postulate, following Cauchy, that we can define a *traction vector* $\boldsymbol{t}$ such that

$$d\boldsymbol{f} = \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})\, ds\ . \tag{3.33}$$

Cauchy proved that there exists a unique symmetric second-order tensor $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ (i.e,$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, the physical meaning of which will be discussed later) such that

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) = \boldsymbol{\sigma}(\boldsymbol{x}, t)\, \boldsymbol{n}\ . \tag{3.34}$$

The spatial tensor $\boldsymbol{\sigma}$ is called the *Cauchy stress*. Its physical meaning becomes clear when we write Eq. (3.34) in components form, $t_i = \sigma_{ij} n_j$. Therefore, σ_{ij} is the force per unit area in the ith direction, acting on a surface element whose outward normal has a component n_j in the jth direction. A corollary of Eq. (3.34)

$$\boldsymbol{t}(\boldsymbol{x}, t, -\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) \ , \tag{3.35}$$

is nothing but *Newton's third law (action and reaction).*

As $\boldsymbol{\sigma}$ is defined in terms of the deformed configuration, which is not known a priori (one should solve for it using the stresses themselves), $\boldsymbol{\sigma}$ is not always a useful quantity (it is the only relevant quantity in the linearized field theory of elasticity, where we do not distinguish between the deformed and undeformed configurations). To overcome this difficulty, we can define alternative stress measures that are useful for calculations. In general, we will show later that thermodynamics allows us to define for any strain measure a work-conjugate stress measure. Here, we define one such mechanically-motivated stress measure. Let us define a (fictitious) reference configuration traction vector $\boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N})$ as

$$d\boldsymbol{f} = \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})\, ds = \boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N})\, dS \ , \tag{3.36}$$

where $\boldsymbol{N}$ and dS are the reference outward normal and unit area, respectively, whose images in the deformed configuration are $\boldsymbol{n}$ and ds, respectively. Following Cauchy, there exists a tensor $\boldsymbol{P}(\boldsymbol{X}, t)$ such that

$$\boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N}) = \boldsymbol{P}(\boldsymbol{X}, t)\, \boldsymbol{N} \ . \tag{3.37}$$

$\boldsymbol{P}(\boldsymbol{X}, t)$ is called the *first Piola-Kirchhoff stress tensor*. In fact, it is not a true tensor (it relates quantities from the deformed and undeformed configuration and hence, like $\boldsymbol{F}$, is a two-point tensor) and is not symmetric. Using the above properties, it is straightforward to show that it is related to the Cauchy stress $\boldsymbol{\sigma}$ by

$$\boldsymbol{P} = J \boldsymbol{\sigma} \boldsymbol{F}^{-T} \ . \tag{3.38}$$

The concepts of strain and stress will allow us to formulate physical laws, such as conservation laws and the laws of thermodynamics, and constitutive laws which describe material behaviors, in the rest of this course.

IV. EQUATIONS OF MOTION, THE LAWS OF THERMODYNAMICS AND OBJECTIVITY

A. Conservation laws

We first consider the mass density in the reference configuration $\rho_0(\boldsymbol{X},t)$. The conservation of mass simply implies that

$$M = \int_{\Omega_0} \rho_0(\boldsymbol{X},t) d\boldsymbol{X}^3 \tag{4.1}$$

is time-independent (Ω_0 is the region occupied by the body in the reference configuration), i.e.,

$$\frac{DM}{Dt} = \frac{D}{Dt}\int_{\Omega_0} \rho_0(\boldsymbol{X},t) d\boldsymbol{X}^3 = \frac{D}{Dt}\int_{\Omega} \rho(\boldsymbol{x},t) d\boldsymbol{x}^3 = 0 \ , \tag{4.2}$$

where Ω is the region occupied by the body in the deformed configuration. The integral form can be easily transformed into a local form. In the reference (Lagrangian) configuration it simply reads

$$\frac{D\rho_0}{Dt} = \frac{\partial \rho_0(\boldsymbol{X},t)}{\partial t} = 0 \quad \Longrightarrow \quad \rho_0(\boldsymbol{X},t) = \rho_0(\boldsymbol{X}) \ . \tag{4.3}$$

To obtain the local form in the Eulerian description, note that (by the definition of J, cf., Eq. (3.29)) $\rho_0(\boldsymbol{X})\!=\!\rho(\boldsymbol{x},t)J(\boldsymbol{X},t)$ and $\dot{J}\!=\!J\,\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v}$ (prove). Therefore,

$$\frac{D\rho_0}{Dt} = \frac{D}{Dt}\Big[\rho(\boldsymbol{x},t)J(\boldsymbol{X},t)\Big] = J\frac{D\rho}{Dt} + J\rho\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} = 0 \ , \tag{4.4}$$

which implies

$$\frac{D\rho(\boldsymbol{x},t)}{Dt} + \rho(\boldsymbol{x},t)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v}(\boldsymbol{x},t) = \frac{\partial\rho(\boldsymbol{x},t)}{\partial t} + \nabla_{\boldsymbol{x}}\cdot\Big(\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\Big) = 0 \ . \tag{4.5}$$

This expression of local mass conservation (continuity equation) takes the general form of a local conservation law

$$\frac{\partial(\text{field})}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\text{field flux}) = \text{source/sink} \ . \tag{4.6}$$

Let us now discuss a theorem that will be very useful in formulating and manipulating other conservation laws. Consider the following 1D integral involving an Eulerian scalar field $\psi(x,t)$

$$I(t) = \int_{x_1=\varphi(X_1,t)}^{x_2=\varphi(X_2,t)} \psi(x,t)dx \ . \tag{4.7}$$

Note that $X_{1,2}$ are fixed here. Taking the time derivative of $I(t)$ (Leibnitz's rule) we obtain

$$\dot{I}(t) = \int_{\varphi(X_1,t)}^{\varphi(X_2,t)} \partial_t\psi(x,t)dx + \psi\Big(\varphi(X_2,t),t\Big)\partial_t\varphi(X_2,t) - \psi\Big(\varphi(X_1,t),t\Big)\partial_t\varphi(X_1,t) \ . \tag{4.8}$$

First recall that (generally in 3D)

$$\boldsymbol{V}(\boldsymbol{X},t) = \partial_t \boldsymbol{\varphi}(\boldsymbol{X},t) = \boldsymbol{v}(\boldsymbol{x},t) \ . \tag{4.9}$$

Then note that since X_1 and X_2 are fixed in the integral we can interpret the time derivative as a material time derivative D/Dt. Therefore, we can rewrite Eq. (4.8) as

$$\frac{D}{Dt}\int_{\varphi(X_1,t)}^{\varphi(X_2,t)} \psi(x,t)dx = \int_{\varphi(X_1,t)}^{\varphi(X_2,t)} \Big[\partial_t\psi(x,t) + \partial_x\Big(\psi(x,t)v(x,t)\Big)\Big]\, dx \ . \tag{4.10}$$

The immediate generalization of this result to volume integrals over a time dependent domain Ω reads

$$\frac{D}{Dt}\int_{\Omega} \psi(\boldsymbol{x},t)d\boldsymbol{x}^3 = \int_{\Omega} \Big[\partial_t\psi(\boldsymbol{x},t) + \nabla_{\boldsymbol{x}}\cdot\Big(\psi(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\Big)\Big]\, d\boldsymbol{x}^3 \ . \tag{4.11}$$

This is the Reynolds' transport theorem which is very useful in the context of formulating conservation laws. This is the same Osborne Reynolds (1842-1912), who is known for his studies of the transition from laminar to turbulent fluid flows, and who gave the Reynolds number its name.

Using mass conservation, we obtain (prove)

$$\frac{D}{Dt}\int_{\Omega} \rho(\boldsymbol{x},t)\psi(\boldsymbol{x},t)d\boldsymbol{x}^3 = \int_{\Omega} \rho(\boldsymbol{x},t)\frac{D\psi(\boldsymbol{x},t)}{Dt}d\boldsymbol{x}^3 \ . \tag{4.12}$$

This is very useful when we choose $\psi(\boldsymbol{x},t)$ to be a quantity per unit mass. In particular, setting $\psi\!=\!1$ we recover the conservation of mass.

Linear momentum balance (Newton's second law) reads

$$\dot{\boldsymbol{P}}(t) = \frac{D}{Dt}\int_{\Omega_0} \rho_0(\boldsymbol{X})\boldsymbol{V}(\boldsymbol{X},t)d\boldsymbol{X}^3 = \frac{D}{Dt}\int_{\Omega} \rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)d\boldsymbol{x}^3 = \boldsymbol{F}(t) \ , \tag{4.13}$$

where $\boldsymbol{F}(t)$ is the total force acting on a volume element Ω (do not confuse $\boldsymbol{P}$ with the first Piola-Kirchhoff stress tensor of Eq. (3.37)). To obtain a local form of this law note that the total force is obtained by integrating local tractions (surface forces) $\boldsymbol{t}(\boldsymbol{x},t)$ and body (volume) forces $\boldsymbol{b}(\boldsymbol{x},t)$, i.e.,

$$\boldsymbol{F}(t) = \int_{\partial\Omega} \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})ds + \int_{\Omega} \boldsymbol{b}(\boldsymbol{x},t)d\boldsymbol{x}^3 \ , \tag{4.14}$$

where $\partial\Omega$ is the boundary of the volume element. Use Cauchy's stress theorem of Eq. (3.34) and the divergence (Gauss) theorem of Eq. (2.27) to obtain

$$\int_{\partial\Omega} \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})ds = \int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}ds = \int_{\Omega} \nabla_{\boldsymbol{x}}\cdot\boldsymbol{\sigma}(\boldsymbol{x},t)d\boldsymbol{x}^3 \ . \tag{4.15}$$

Use then Reynold's transport theorem of Eq. (4.12), with ψ replaced by the spatial velocity field $\boldsymbol{v}$, to transform the linear momentum balance of Eq. (4.13) into

$$\int_{\Omega} \left[\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\sigma}(\boldsymbol{x},t) + \boldsymbol{b}(\boldsymbol{x},t) - \rho(\boldsymbol{x},t)\dot{\boldsymbol{v}}(\boldsymbol{x},t)\right] d\boldsymbol{x}^3 = 0 \ . \tag{4.16}$$

Since this result is valid for an arbitrary material volume, we obtain the following spatial (Eulerian) local form of linear momentum conservation

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \rho\, \dot{\boldsymbol{v}} = \rho \left(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v}\right) \ . \tag{4.17}$$

Note that this equation does not conform with the structure of a general conservation law in Eq. (4.6). This can be achieved (prove), yielding

$$\frac{\partial(\rho\, \boldsymbol{v})}{\partial t} + \nabla_{\boldsymbol{x}} \cdot \left(\boldsymbol{\sigma} - \rho\, \boldsymbol{v} \otimes \boldsymbol{v}\right) = \boldsymbol{b} \ . \tag{4.18}$$

A similar analysis can be developed for the angular momentum. However, the requirement that the angular acceleration remains finite implies that angular momentum balance, at the continuum level, is satisfied if the Cauchy stress tensor $\boldsymbol{\sigma}$ is symmetric, i.e.,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \ , \tag{4.19}$$

to be derived in the tutorial. We note that the symmetry of the Cauchy stress tensor emerges from the conservation of angular momentum if the continuum assumption is valid at all lengthscales. Real materials, however, may possess intrinsic lengthscales associated with their microstructure (e.g., grains, fibers and cellular structures). In this case, we need generalized theories which endow each material point with translational and rotational degrees of freedom, describing the displacement and rotation of the underlying microstructure. One such theory is known as Cosserat (micropolar) continuum, which is a continuous collection of particles that behave like rigid bodies. Under such circumstances one should consider a couple-stress tensor (which has the dimensions of stress $\times$ length) as well, write down an explicit angular momentum balance equation and recall that the ordinary stress tensor is no longer symmetric.

The local momentum conservation laws can be expressed in Lagrangian forms. For example, the linear momentum balance, Eq. (4.17), translates into (prove)

$$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{P} + \boldsymbol{B} = \rho_0 \dot{\boldsymbol{V}} \ , \tag{4.20}$$

where $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor of Eq. (3.37) and $\boldsymbol{B}(\boldsymbol{X}, t) = J(\boldsymbol{X}, t)\boldsymbol{b}(\boldsymbol{x}, t)$. This equation is extremely useful because it allows calculations to be done in a fixed undeformed coordinate system $\boldsymbol{X}$. It is important to note that one should also transform the boundary conditions of a given problem from the deformed configuration (where they are physically imposed) to the underformed configuration.