# Wind Waves Generation 

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# Wind Waves Generation 

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To my family

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## Abstract

This study deals with the problem of wind waves generation. Scientific interest in momentum and energy transfer between the ocean and atmosphere and wave forecasting are examples of directly related research topics of the wind waves generation problem. The problem deals with the instability of water waves in the presence of a shear flow. The study contains a full formulation of the linear stability problem in $2 D$ for the viscous and inviscid models. The formulation leads to an ODE which controls the problem. The governing equation for the inviscid model is Rayleigh's equation, whereas the governing equation for the viscous model is the Orr-Sommerfeld equation. After applying the boundary conditions the resulting problem is an eigenvalue problem for the wavenumber or for the wave frequency. These eigenvalue problems were solved using numerical methods chosen especially for each model. The mean flow of the air and water plays a main role in the problem because the solution is sensitive to this choice. We use three versions of the mean flow profile; two of them are profiles which have been used in previous studies and one of them is a new profile which we suggest as a more physical profile. The results were calculated for both models and many different scenarios. In the viscous model, we expand the range of wavelengths and wind intensities with respect to previous studies to $0.001 \mathrm{~m}<\lambda<0.2 \mathrm{~m}, 0.1 \mathrm{~m} / \mathrm{sec}<u_{*}<1 \mathrm{~m} / \mathrm{sec}$. The results are presented in a comprehensive set of figures. In the viscous model we discovered the presence of a new unstable mode at high wind intensities for the case in which we used a profile with a shear current. This second unstable mode is characterized by a slower phase velocity. A comparison between the results of the inviscid model and the viscous model is of major important. We compare these two models not only by comparing the eigenvalues and the eigenfunctions, but also by comparing the pattern of the dynamic boundary condition. The comparison at low wind intensities shows that the inviscid model and the viscous model have similar patterns, although the growth rates
in the viscous model are twice as large or more than those of the inviscid model. At high wind intensities the results of these two models are far from similar. The results of the comparison emphasize the question, in what sense is Rayleigh's equation is an approximation for a large Reynolds number to the Orr-Sommefeld equation.

## List of symbols

| $x$ | Horizontal coordinate |
| :--- | :--- |
| $z$ | vertical coordinate |
| $t$ | Time coordinate |
| $U$ | Mean flow profile |
| $u$ | Perturbation of the horizontal velocity |
| $v$ | Perturbation of the vertical velocity |
| $u_{*}$ | Friction velocity |
| $p$ | Perturbation of the pressure |
| $k$ | Wavenumber |
| $\lambda$ | Wavelength |
| $\omega$ | Wave frequency |
| $T$ | Waveperiod |
| $c$ | Phase velocity |
| $c_{g}$ | Group velocity |
| $\rho$ | Density |
| $\mu$ | Dynamic viscosity |
| $\nu$ | Kinematic viscosity |
| $\sigma$ | Surface tension |

$g \quad$ Acceleration of gravity
$\kappa \quad$ Von-Karman's constant
$\alpha \quad$ Spatial growth rate
$\beta$ Temporal growth rate
$\eta \quad$ Water surface curve
$f$ Auxiliary function
$\psi \quad$ Stream function
$\tau \quad$ Stress
$z_{0} \quad$ Roughness
$z_{\infty} \quad$ The value of the finite domain
$T_{n} \quad$ The $n t h$ Chebyshev polynomial
$\alpha_{c h}$ Charnock's constant
$R \quad$ Reynolds number
$F \quad$ Inverse square Froude number
$W \quad$ Inverse Weber number
$X_{0} \quad$ Index zero - reference problem
$X_{a} \quad$ Index a - air quantities
$X_{w} \quad$ Index a - water quantities

## Chapter 1

## Introduction

### 1.1 Description of the problem and motivation

Human kind's interest in the behavior of water waves began when man started to settle along the coasts and travel the oceans. The interest in the generation mechanisms of waves was a challenging problem for scientists since ancient times. The motivation for a better understanding of this complex phenomenon comes from many disciplines. The problem of water wave generation by wind basically deals with the interaction between two fluids: one is a liquid and the other a gas; Together these two fluids generate a coupled system. Such systems exist in nature and there are additional scenarios of coupled systems, other than ones including water and air. For example, there can be oil instead of water or even waves at other planets when the liquid is hydrogen and the gas is the planet atmosphere. Generally, the problem deals with the transport of momentum and energy between the atmosphere and the ocean. This knowledge is important in order to estimate the mixing of the upper layer of the ocean. Another important process which is related to this subject is gas dissolution in the ocean; For instance, the gas can be $\mathrm{CO}_{2}$ (which is a greenhouse gas).

In our problem the air is above the water, as there is a wavy interface between the two fluids. There is a velocity field in the air which can be divided into two components. The main component is the mean flow-which is the wind, and the secondary component is the flow due to the waves. The assumption is that the mean wind is a parallel flow and the nature of the wind depends on the specific scenario. There can be


Figure 1.1: Schematic description of the problem תאור סכמטי של הבעיה
a similar division for the velocity field in the water where the main component is the current, and the secondary component is the flow due to the waves. The mean current is assumed to be a parallel flow. In the more specific problem which this study deals with, the water depth, as well as the air layer thickness, are infinite. Between the atmosphere and the ocean there is momentum and energy exchange. For example, when the wind starts to blow above an Slightly wavy interface it loses energy and momentum to the water. This momentum and energy produce waves and current, which in case of energy transfer can be dissipated when the waves are breaking. The opposite scenario is when waves travel against the wind; The water can lose energy and momentum and thus the waves will decay.

### 1.2 Literature survey

In this section we will survey only the publications which we think to be important to our study, since throughout the years there has been an enormous number of publications on the subject. The discussion of wave generation in modern science is almost 150
years old, since the days of Kelvin, Stokes, Rayleigh and others. Since ancient times, men began to understand that water waves are generated by the wind. The history of scientific publications on the subject in the 20th century started in 1925 when Jeffreys published his sheltering theory [10]. He suggested that the mechanism of wave generation is such in which there is some kind of separation in the air flow above the waves. This separation occurs somewhere in the leeward side of the wave crest. Such a separation produces a phase shift between the wave shape signal and the pressure signal. Jeffreys also showed that such a phase shift can do work on the water and formulated the energy flux as:

$$
\begin{equation*}
\frac{\partial E}{\partial t}=\overline{p(z=\eta) \frac{\partial \eta}{\partial t}} \tag{1.2.1}
\end{equation*}
$$

Where $E$ is energy per area unit, $t$ is time, $p$ is the air pressure and $\eta$ is the interface curve. The bar denotes averaging with respect to time over the wave period. Major progress was made in 1957 with the publication of two groundbreaking studies by Miles [14] and Phillips [18]. These two studies suggest two different mechanisms for the wave generation. Phillips argued that waves can be generated by a resonance mechanism between the air turbulent eddies and the water [18]. In his study he assumed that the water is an inviscid fluid and the initial water state is rest. Miles proposed that the growth of waves is caused by interaction of the surface waves with a parallel shear flow [14]. In his first article he assumed that the fluids are inviscid and presented Rayleigh's equation as the governing equation of the problem.

$$
\begin{equation*}
(U-c)\left(\phi^{\prime \prime}-k^{2}\right)-\phi U^{\prime \prime}=0 \tag{1.2.2}
\end{equation*}
$$

He also argued that the rate of energy transfer to a wave of speed $c$ is proportional to the wind profile curvature $-U_{a}^{\prime \prime}\left(z_{c r}\right)$ at the critical height, where $U_{a}(z)=c$. In 1959 Miles published a further article [16]. In this article he studied aspects such as: imposing the boundary conditions at the water surface rather than at the mean surface, including the viscosity and applying more accurate numerical calculations. His results showed that imposing the boundary conditions at a more accurate location has no effect, the viscosity has a very small effect in gravity waves and the more accurate numerical results estimate smaller growth rate than his previous study. Miles actually used a perturbation model when he assumed that the wave amplitude is infinitesimal and the flow due to the waves is laminar. In his model there is a background velocity (i.e. mean flow profile). This profile depends on the specific case, but since the assumption that
the air flow is turbulent is very common this mean flow profile is often taken as an approximation of the turbulent boundary layer equation. Phillips and Miles suggest different approaches in order to estimate the influence of wind on water waves. These different approaches influence the many publications which came later. Many authors after Phillips have studied the effect of small scale turbulence on wave growth.

Chalikov [5] published a model in 1976 which deals with the growth of a single wave and the effects of nonlinearity, as well as the structure of airflow over a spectrum of waves and the effects of atmospheric stability. A further study by Chalikov and Makin [6] contains a determination of the drag coefficient over water waves. More complicated turbulent models were advocated by Gent and Taylor [8]. The former uses one equation model, whereas the latter uses a two equation model. These studies deal with a numerical calculation of the wave growth due to a Phillips-like mechanism because of the complexity of the problem. However Jacobs [9] uses a simple eddy-viscosity model and obtains a very elegant expression of the growth rate by means of a matched asymptotic expansion, where the small parameter is the drag coefficient. Valenzuela [24] made a comprehensive numerical study of the growth of gravity-capillary waves. He adopted Miles' approach and solved the coupled air-water stability problem for two viscous fluids in by a shear flow. Valenzuela uses finite-difference methods in order to transform the problem to an algebraic eigenvalue problem.

Valenzuela uses the lin-log profile in both media as the base flow and shows that the shear flow in the water can not be ignored. Kawai [11] investigates the generation of initial wavelets, and combines experimental and theoretical studies. Kawai's main interest is in the most unstable wave which can grow under a specific friction velocity. Kawai uses a lin-log wind profile and an error-function-like current Profile in his calculations. The numerical solution that Kawai uses is based on an integration of the Orr-Sommerfeld equation using Runge-Kutta method with a purifying procedure in order to keep the solution stable. He argues that the linear instability mechanism controls the process of wave generation for the first 10sec. Van-Gastel et al. [26] study the effect of wind on gravity-capillary wind waves using asymptotic methods. They also solve a pair of Orr-Sommerfeld's equations. They argue that in the growth of the initial wavelets, the first wave to be generated is proportional to $u_{*}^{3}$. Van-Gastel et al. also study the effect of the wind and current profile shape, and find that the growth rate
is very sensitive to the wind profile shape; the influence of the current shape is much smaller, but the drift current and the shear current at the interface have a great influence on the phase velocity. Wheless and Csanady [27] use compound matrix methods in order to integrate the Orr-Sommerfeld equation and investigate the stability of short waves. In their calculations they use an error-function-like wind profile and an exponential current profile. They also study the effects of wind profile on the growth rate, and argue that the surface tension has less influence; the growth rate increases when the surface tension decreases. Wheless and Csanady try to study the meaning of the eigenfunction vertical distribution and argue that the perturbation vorticity is high; the streamwise surface velocity perturbation in typical cases can be five times the orbital velocity of free waves on undisturbed water surface. Hence this suggests that unstable waves should therefore be thought of as a fundamentally different flow structure from free waves. Boomkamp et al. [3] solve the problem of waves on a thin film of liquid sheared by gas, which is a very similar problem. Boomkamp et al. use the Chebyshev collocation method for solving the stability problem. They show a robust method which converges easily for many cases and that is easy to apply. Tsai Grass and Simon [23] study the spatial growth of gravity-capillary waves sheared by laminar air flow, using experimental and theoretical tools. They use a fourth order Runge-Kutta method to integrate the Orr-Sommerfeld equation and some kind of filtering scheme in order to remove parasitic errors. They use a Lock-like profile for both the wind and the current. Zhang [28] studies the effects of shear flow on the stability of short surface waves. He uses an inviscid model and as a result uses Rayleigh's equation as the governing equation. Zhang proposes a new method in order to get an approximate solution. The method which is called piecewise linear approximation (PLA), approximates the wind and current profile as linear in a specific segment, and thus the curvature is zero and the equation has a very simple form. A special segment is when it contains the critical point where $U\left(z_{c}\right)=c$, and Zhang gives this segment a special treatment. Stiassnie Agnon and Janssen [21] also study the instability of water waves where the fluids are assumed to be inviscid. They use the so-called regular approach in order to pass through the critical point when integrating the Rayleigh's equation. The results of their calculation are similar to those of previous calculations for $\frac{u_{*}}{c_{0}}<2$, for $\frac{u_{*}}{c_{0}}>2$ they obtain a maximum in the growth rate, which does not appear in previous studies. Stiassnie et al.
study the difference between the temporal and the spatial case and find that the ratio of growth rates $\frac{\alpha}{\beta}$ deviate up to $20 \%$ from the leading order of value of 2 .

Shemdin and Yun [20] try to validate Miles' theory by experimenting. They measure aerodynamic pressure over mechanically generated water waves. They use a pressure sensor that follows the water surface. Shemdin and Yun show that there is a significant phase shift between the pressure signal and the water surface, and that this phase shift grows when increasing the wind intensity. Larson and Wright [13] did a comprehensive experimental study of the temporal growth of gravity-capillary waves. They used microwave backscatter as a measurement technique. Larson and Wright find that the growth rate is independent of the fetch, dependent on the wavenumber and varying with $u_{*}$ like a power law $\beta=f(k) u_{*}^{n}$ where $n \approx 1.484$. They also find that the growth rate has a maximum near $\lambda_{\text {min }} \approx 1.73 \mathrm{~cm}$. Banner and Melville [2] experimentally investigate the flow separation over water waves. They generate short gravity waves and use visualization methods to show the separation in a breaking wave. They argue that the condition for separation, which is a stagnation point, can occur only for the case of a breaking wave. They also study how this separation affects the air mean flow profile. Kawai [11] in his studies also experimented with gravity-capillary waves and tried to show a correlation between the theoretical results and the experimental results. He looked for the first wave to be generated at a given friction velocity $u_{*}$. Kawai also measured the flow in the water and the drift current, and showed the evolution of the current with time. Kawai used a resistance type wave-gauge to measure the wave growth and a shadowgraph-photography to measure the phase velocity. Mitsuyasu and Honda [17] measured spatial growth of mechanically generated gravity waves $0.6 s e c<T<1.3 s e c, 0.95 m<\lambda_{0}<2.63 m$. They found that the mechanically generated waves grew exponentially under the action of the wind. They transfered the spatial growth rate to temporal growth rate through $\beta=C_{q} \alpha$ and proposed a new empirical formula for the growth rate in the range $0.1<\frac{u_{*}}{c}<1$. Mitsuyasu and Honda also studied the effects of wave steepness $H / L$ and argued that it seems that this effect is small. Caulliez Ricci and Dupont 1998 [4] experimentally study the first visible ripples that appear on the water surface. They argue that the laminar-turbulent transition of the near surface water flow causes an explosive growth of the wind generated ripples. These ripples become visible and thus mark the surface of the well localized

V-shape turbulent zone forming the streaks. As previously mentioned Tsai et al. [23], also performed experimental studies of the spatial growth of gravity-capillary waves. They measure a laminar wind profile at varying fetches and emphasize the development of the boundary layer with fetch. The experimental procedure they used was a wind-tunnel wave tank in-which they could produce the laminar profile and then use a twin laser beam technique to measure the wave properties. The waves were mechanically generated waves with $k a \sim 10^{-3}$. They argue that there is a good correlation between the experimental and numerical results and hence this linear stability mechanism determines the initial stage of wave growth. In all these articles the subject of the wind velocity profile is a dominant issue and sometimes it seems as though the number of authors is equal to the number of profile versions.

Charnock [7] measures the air mean velocity profile above a large reservoir. He finds that the air flow fits a logarithmic law according to:

$$
\begin{equation*}
\frac{u}{u_{*}}=\frac{1}{k} \log \frac{z}{z_{0}} \tag{1.2.3}
\end{equation*}
$$

Charnock obtains that $\frac{g z_{0}}{u_{*}^{2}}=$ constant and suggests that the wind profile over the water surface will be:

$$
\begin{equation*}
\frac{u}{u_{*}}=\frac{1}{k} \log \frac{g z}{u_{*}^{2}}+\text { constant } \tag{1.2.4}
\end{equation*}
$$

Miles [15] suggests an approximation to the solution of the boundary layer equation which has a linear zone and a logarithmic-like profile. This profile was very useful for many authors because of its smooth first derivative for all values of $u_{*}$ and matching height between the linear and the logarithmic regions. Most of the field measurements were made at a low wind speed. As evidence of the nature of air flow above water at high speed wind, we can cite Powell et al. [19] who made field measurements in tropical cyclones. They found that the wind profile correlates very well to the logarithmic shape at the first 200 m . By determining surface stress roughness length and natural stability drag coefficient, they found that the surface momentum flux levels off as the wind speed increases above hurricane force.

## Chapter 2

## Mathematical Formulation of the

## Problem

### 2.1 Orr-Sommerfeld equation

The starting point is the governing equations of an incompressible viscous fluid flow neglecting thermal effects, which are the Navier-Stokes and the continuity equations.

$$
\begin{gather*}
\rho\left(\frac{\partial \vec{V}}{\partial t}+\vec{V} \cdot \vec{\nabla} \vec{V}\right)=-\vec{\nabla} P+\mu \nabla^{2} \vec{V}+\vec{g} \rho  \tag{2.1.1}\\
\vec{\nabla} \cdot \vec{V}=0 \tag{2.1.2}
\end{gather*}
$$

Define the velocity field and the pressure field as:

$$
\begin{gather*}
\vec{V}=(U(z)+u(x, z, t), v(x, z, t))  \tag{2.1.3}\\
P=p_{0}-\rho g z+p(x, z, t) \tag{2.1.4}
\end{gather*}
$$

Where $U(z)$ is the base flow, $u, v$ and $p$ are harmonic perturbations of the horizontal velocity, vertical velocity and pressure, respectively. We can separate the equations into harmonic terms and steady terms. Under the assumption that the perturbations are infinitesimal, we can linearize the equations and receive the following system.

Horizontal momentum:

$$
\begin{equation*}
\rho\left(\frac{\partial u}{\partial t}+U \frac{\partial u}{\partial x}+v U^{\prime}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial z^{2}}\right) \tag{2.1.5}
\end{equation*}
$$

Vertical momentum:

$$
\begin{equation*}
\rho\left(\frac{\partial v}{\partial t}+U \frac{\partial v}{\partial x}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^{2} v}{\partial x^{2}}+\frac{\partial^{2} v}{\partial z^{2}}\right) \tag{2.1.6}
\end{equation*}
$$

Continuity:

$$
\begin{equation*}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}=0 \tag{2.1.7}
\end{equation*}
$$

Since we are looking for a solution that has a harmonic part and a height dependent part, it has the form:

$$
\begin{equation*}
u=\tilde{u}(z) e^{i(k x-\omega t)}, \quad v=\tilde{v}(z) e^{i(k x-\omega t)}, \quad p=\tilde{p}(z) e^{i(k x-\omega t)}, \quad \eta=\eta_{0} e^{i(k x-\omega t)} \tag{2.1.8}
\end{equation*}
$$

We can define the stream function which has the form:

$$
\begin{equation*}
\psi=f(z) e^{i(k x-\omega t)} \tag{2.1.9}
\end{equation*}
$$

Where $\omega, k$ are complex numbers, and $\tilde{u}, \tilde{v}, \tilde{p}$ are of course complex functions. Hence:

$$
\begin{equation*}
\tilde{v}=-i k f(z), \quad \tilde{u}=f^{\prime}(z) \tag{2.1.10}
\end{equation*}
$$

Substitute into (2.1.5):

$$
\begin{equation*}
\rho\left(-i \omega f^{\prime}+i k U f^{\prime}-i k U^{\prime} f\right)=-i k \tilde{p}+\mu\left(-k^{2} f^{\prime}+f^{\prime \prime \prime}\right) \tag{2.1.11}
\end{equation*}
$$

The expression for the pressure is:

$$
\begin{equation*}
\tilde{p}=\rho U^{\prime} f+(i k \mu-\rho(U-c)) f^{\prime}-\frac{i \mu}{k} f^{\prime \prime \prime} \tag{2.1.12}
\end{equation*}
$$

Finding the derivative of $\tilde{p}$ :

$$
\begin{equation*}
\tilde{p}^{\prime}=\left[\mu\left(f^{(4)}-k^{2} f^{\prime \prime}\right)-\rho\left(i \omega f^{\prime \prime}+i k\left(U f^{\prime \prime}-f U^{\prime \prime}\right)\right)\right] \frac{1}{i k} \tag{2.1.13}
\end{equation*}
$$

Substituting it into (2.1.6) finally yields the Orr-Sommerfeld equation (2.1.14):

$$
\begin{equation*}
i \nu\left(f^{(4)}-2 k^{2} f^{\prime \prime}+k^{4} f\right)+k\left[(U-c)\left(f^{\prime \prime}-k^{2} f\right)-U^{\prime \prime} f\right]=0 \tag{2.1.14}
\end{equation*}
$$

Until this point we did not discuss the specific problem of wave generation; this formulation is valid for all incompressible viscous fluids with infinitesimal harmonic perturbations. In the specific problem of wave generation we have two fluids; in most cases one is liquid (water) and the other is a gas (air). Both the water and the air should satisfy the Orr-Sommerfeld equation. We denote the different media by subindex $X_{a}, X_{w}$ for air and water, respectively.

Boundary conditions: The domain of the problem is $z \in[-\infty, 0]$ for the water, and $z \in[0, \infty]$ for the air. There are two kinds of boundary conditions: The first is the boundary condition at infinity and the second is the interface boundary conditions. At infinity we want the perturbations' amplitude to vanish. Asymptotically, the equation has four independent exponential solutions; we choose only the decayed solutions. The dominant asymptotic solution in the water or in the air is $e^{ \pm k z}$, respectively; hence the boundary conditions at infinity are:

$$
\begin{gather*}
f_{w}(z) e^{-k z}=\text { const, } z \rightarrow-\infty  \tag{2.1.15}\\
f_{a}(z) e^{k z}=\text { const }, z \rightarrow \infty \tag{2.1.16}
\end{gather*}
$$

On the interface we need to satisfy the following conditions: kinematic boundary condition, continuity of vertical and horizontal velocity, continuity of shear stress and the dynamic boundary condition (continuity of normal stress). In the boundary condition we also refer to the harmonic terms only. Most of the boundary conditions are nonlinear because the interface curve $z=\eta(x, t)$ is also unknown. We use the assumption of infinitesimal perturbation in order to linearize the boundary conditions and we use Taylor expansion about $z=0$, which is the unperturbed interface and uses only the linear terms under the assumption that $\eta$ is infinitesimal. Note that the right sequence of the linearized process is to write the Taylor expansion and then to linearize the condition.

The kinematic boundary condition is:

$$
\begin{equation*}
\frac{\partial \eta}{\partial t}+\vec{V}_{\perp} \cdot \vec{\nabla}_{\perp} \vec{V}=v \quad \text { at } z=\eta \tag{2.1.17}
\end{equation*}
$$

This condition is true for both media-water and air. Hence after linearization:

$$
\begin{gather*}
\frac{\partial \eta}{\partial t}+U_{w, a} \frac{\partial \eta}{\partial x}=v_{w, a} \quad \text { at } z=0  \tag{2.1.18}\\
f_{w, a}=\eta_{0}\left(c-U_{0}\right) \text { at } z=0 \tag{2.1.19}
\end{gather*}
$$

Since the problem is linear in terms of the auxiliary functions $f_{w, a}$ and we can see that $f_{w, a}(0)$ and $\eta_{0}$ are proportional, it is comfortable to take $\eta_{0}=1$ and to remember that if we want to get $f_{w, a}$ values we need to multiply it by $\eta_{0}[m]$. This choice is arbitrary and it can not influence the resulting eigenvalues. Thus from this point on:

$$
\begin{equation*}
f_{w, a}(0)=\left(c-U_{0}\right) \tag{2.1.20}
\end{equation*}
$$

Continuity of vertical velocity is trivial as a result of the kinematic boundary condition and has the form:

$$
\begin{equation*}
v_{w}(0)=v_{a}(0) \Rightarrow f_{w}(0)=f_{a}(0) \tag{2.1.21}
\end{equation*}
$$

Continuity of horizontal velocity:

$$
\begin{equation*}
u_{w}+U_{w}=u_{a}+U_{a} \text { at } z=\eta \tag{2.1.22}
\end{equation*}
$$

After linearization:

$$
\begin{gather*}
u_{w}+\eta U_{w}=u_{a}+\eta U_{a} \text { at } z=0  \tag{2.1.23}\\
f_{w}^{\prime}+U_{w}^{\prime}=f_{a}^{\prime}+U_{a}^{\prime} \text { at } z=0 \tag{2.1.24}
\end{gather*}
$$

Continuity of shear stress at the interface:

$$
\begin{gather*}
\tau_{w, x z}=\tau_{a, x z} \text { at } z=\eta  \tag{2.1.25}\\
\mu_{w}\left(\frac{\partial u_{w}}{\partial z}+U_{w}^{\prime}+\frac{\partial v_{w}}{\partial x}\right)=\mu_{a}\left(\frac{\partial u_{w}}{\partial z}+U_{a}^{\prime}+\frac{\partial v_{w}}{\partial x}\right) \text { at } z=\eta \tag{2.1.26}
\end{gather*}
$$

After linearization:

$$
\begin{gather*}
\mu_{w}\left(\frac{\partial u_{w}}{\partial z}+\eta U_{w}^{\prime \prime}+\frac{\partial v_{w}}{\partial x}\right)=\mu_{a}\left(\frac{\partial u_{w}}{\partial z}+\eta U_{a}^{\prime \prime}+\frac{\partial v_{w}}{\partial x}\right) \text { at } z=0  \tag{2.1.27}\\
\mu_{a}\left(f_{a}^{\prime \prime}+k^{2} f_{a}+U_{a}^{\prime \prime}\right)=\mu_{w}\left(f_{w}^{\prime \prime}+k^{2} f_{w}+U_{w}^{\prime \prime}\right) \text { at } z=0 \tag{2.1.28}
\end{gather*}
$$

The dynamic boundary condition (continuity of normal stress) is:

$$
\begin{align*}
\tau_{w, z z} & =\tau_{a, z z}+\sigma \nabla_{\perp}^{2} \eta \text { at } z=\eta  \tag{2.1.29}\\
-P_{w}+2 \mu_{w} \frac{\partial v_{w}}{\partial z} & =-P_{a}+2 \mu_{a} \frac{\partial v_{a}}{\partial z}+\sigma \frac{\partial^{2} \eta}{\partial x^{2}} \text { at } z=\eta \tag{2.1.30}
\end{align*}
$$

After linearization:

$$
\begin{equation*}
p_{w}-\rho_{w} g \eta-2 \mu_{w} \frac{\partial v_{w}}{\partial z}=p_{a}-\rho_{a} g \eta-2 \mu_{a} \frac{\partial v_{a}}{\partial z}-\sigma \frac{\partial^{2} \eta}{\partial x^{2}} \text { at } z=0 \tag{2.1.31}
\end{equation*}
$$

And after substitute of the expression for $p$ :

$$
\begin{array}{r}
\rho_{w}\left[k f_{w}^{\prime}\left(c-U_{0}\right)+k f_{w} U_{w}^{\prime}+i \nu_{w}\left(3 k^{2} f_{w}^{\prime}-f_{w}^{\prime \prime \prime}\right)-g k\right]= \\
=\rho_{a}\left[k f_{a}^{\prime}\left(c-U_{0}\right)+k f_{a} U_{a}^{\prime}+i \nu_{a}\left(3 k^{2} f_{a}^{\prime}-f_{a}^{\prime \prime \prime}\right)-g k\right]+\sigma k^{3} \text { at } z=0 \tag{2.1.32}
\end{array}
$$

The next stage is to normalize the whole formulation. We need to choose characteristic magnitudes. The reference problem is the problem of a linear harmonic wavetrain in infinite depth sea. Hence we will use the values of $\omega_{0}, k_{0}$ as normalizing factors. The
subindex zero will be a symbol for the reference problem. The relation between the frequency and the wavenumber of the reference problem will be:

$$
\begin{equation*}
\omega_{0}^{2}=g k_{0}+\frac{\sigma k_{0}^{3}}{\rho_{w}} \tag{2.1.33}
\end{equation*}
$$

Where $g$ is the acceleration of gravity and $\sigma$ is the surface tension. In the normalizing factor of $f$ we will use $\eta_{0}$ as a perturbation characteristic length.

Define the normalizing procedure (hat symbol=normalizing value):

$$
\begin{gather*}
\hat{\omega}=\frac{\omega}{\omega_{0}}, \hat{k}=\frac{k}{k_{0}}, \hat{c}=\frac{c}{c_{0}} \\
\hat{z}=z k_{0}, \hat{U}=\frac{U}{c_{0}}, \hat{f}=\frac{f k_{0}}{\eta_{0} \omega_{0}} \tag{2.1.34}
\end{gather*}
$$

Before we present the normalized form of the problem we need to define a few dimensionless numbers.

The Reynolds number:

$$
\begin{equation*}
R=\frac{c_{0}}{\nu k_{0}} \tag{2.1.35}
\end{equation*}
$$

The inverse square of Froude number:

$$
\begin{equation*}
F=\frac{1}{F_{r}^{2}}=\frac{g k_{0}}{\omega_{0}^{2}} \tag{2.1.36}
\end{equation*}
$$

The inverse Weber number:

$$
\begin{equation*}
W=\frac{1}{W_{e}}=\frac{\sigma k_{0}^{3}}{\rho_{w} \omega_{0}^{2}} \tag{2.1.37}
\end{equation*}
$$

Note that:

$$
\begin{equation*}
W+F=\frac{1}{\omega_{0}^{2}}\left(g k_{0}+\frac{\sigma k_{0}^{3}}{\rho_{w}}\right)=1 \tag{2.1.38}
\end{equation*}
$$

The ratio of densities and ratio of viscosities (where $\mu=\nu \rho$ ):

$$
\begin{equation*}
\rho=\frac{\rho_{a}}{\rho_{w}}, \quad \mu=\frac{\mu_{a}}{\mu_{w}} \tag{2.1.39}
\end{equation*}
$$

The whole mathematical problem after normalization is given by: (All of the quantities are now normalized and the hat symbols are removed).

The ODEs:

$$
\begin{align*}
& i R_{w}^{-1}\left(f_{w}^{(4)}-2 k^{2} f_{w}^{\prime \prime}+k^{4} f_{w}\right)+k\left[\left(U_{w}-c\right)\left(f_{w}^{\prime \prime}-k^{2} f_{w}\right)-U_{w}^{\prime \prime} f_{w}\right]=0 \quad z \in[-\infty, 0]  \tag{2.1.40}\\
& i R_{a}^{-1}\left(f_{a}^{(4)}-2 k^{2} f_{a}^{\prime \prime}+k^{4} f_{a}\right)+k\left[\left(U_{a}-c\right)\left(f_{a}^{\prime \prime}-k^{2} f_{a}\right)-U_{a}^{\prime \prime} f_{a}\right]=0 \quad z \in[0, \infty] \tag{2.1.41}
\end{align*}
$$

Boundary conditions:

$$
\begin{gather*}
f_{a}(0)=f_{w}(0)=c-U_{0}  \tag{2.1.42}\\
f_{w}^{\prime}+U_{w}^{\prime}=f_{a}^{\prime}+U_{a}^{\prime} \text { at } z=0  \tag{2.1.43}\\
\mu\left(f_{a}^{\prime \prime}+k^{2} f_{a}+U_{a}^{\prime \prime}\right)=\left(f_{w}^{\prime \prime}+k^{2} f_{w}+U_{w}^{\prime \prime}\right) \text { at } z=0  \tag{2.1.44}\\
k f_{w}^{\prime}\left(c-U_{0}\right)+k f_{w} U_{w}^{\prime}+i R_{w}^{-1}\left(3 k^{2} f_{w}^{\prime}-f_{w}^{\prime \prime \prime}\right)-k F= \\
=\rho\left[k f_{a}^{\prime}\left(c-U_{0}\right)+k f_{a} U_{a}^{\prime}+i R_{a}^{-1}\left(3 k^{2} f_{a}^{\prime}-f_{a}^{\prime \prime \prime}\right)-k F\right]+W k^{3} \text { at } z=0  \tag{2.1.45}\\
f_{w}(z) e^{-k z}=\text { const, } z \rightarrow-\infty  \tag{2.1.46}\\
f_{a}(z) e^{k z}=\text { const, } z \rightarrow \infty \tag{2.1.47}
\end{gather*}
$$

### 2.2 Rayleigh's equation

Rayleigh's equation is the governing equation for the inviscid case. We can formulate this case as a special case of the previous one if we set $\mu=0, R \rightarrow \infty$. This of course will cause changes in the boundary conditions as well. The governing equations of this case are the Euler equation and the continuity equation.

$$
\begin{gather*}
\rho\left(\frac{\partial \vec{V}}{\partial t}+\vec{V} \cdot \vec{\nabla} \vec{V}\right)=-\vec{\nabla} P+\vec{g} \rho  \tag{2.2.1}\\
\vec{\nabla} \cdot \vec{V}=0 \tag{2.2.2}
\end{gather*}
$$

The definition of the velocity and pressure fields is the same as in (2.1.3),(2.1.4). Under the assumption of infinitesimal perturbations we can linearize the equations. Define $u, v, p, \eta, \psi$ same as in(2.1.9),( 2.1.8). Now we repeat the process of elimination of the pressure from the horizontal momentum equation, find the derivative $\tilde{p}^{\prime}$ and substitute it into the vertical momentum equation and finally obtain the so-called Rayleigh's equation.

$$
\begin{equation*}
(U-c)\left(f^{\prime \prime}-k^{2} f\right)-U^{\prime \prime} f=0 \tag{2.2.3}
\end{equation*}
$$

Boundary conditions: As in the Orr-Sommerfeld formulation, we have boundary conditions at infinity and the interface boundary conditions. At infinity the boundary conditions are the same as in the viscous problem; the asymptotic solution is $e^{ \pm k z}$ as well. Hence it is the same as in (2.1.15),(2.1.16). In the interface the solution should satisfy only the kinematic and dynamic boundary conditions. The boundary conditions
are nonlinear because $\eta(x, t)$ is unknown. We use the same procedure as in the viscous case in order to linearize the equation and the boundary conditions. The kinematic boundary condition is exactly the same as (2.1.19). We choose again $\eta_{0}=1$ in order to get (2.1.20). The dynamic boundary condition is similar to the viscous case, but without the viscous terms.

$$
\begin{align*}
\tau_{w, z z} & =\tau_{a, z z}+\sigma \nabla_{\perp}^{2} \eta \text { at } z=\eta  \tag{2.2.4}\\
-P_{w} & =-P_{a}+\sigma \frac{\partial^{2} \eta}{\partial x^{2}} \text { at } z=\eta \tag{2.2.5}
\end{align*}
$$

After linearization:

$$
\begin{equation*}
p_{w}-\rho_{w} g \eta-=p_{a}-\rho_{a} g \eta-\sigma \frac{\partial^{2} \eta}{\partial x^{2}} \text { at } z=0 \tag{2.2.6}
\end{equation*}
$$

And after substitution of the expression for $p$ :

$$
\begin{equation*}
\rho_{w}\left[k f_{w}^{\prime}\left(c-U_{0}\right)+k f_{w} U_{w}^{\prime}-g k\right]=\rho_{a}\left[k f_{a}^{\prime}\left(c-U_{0}\right)+k f_{a} U_{a}^{\prime}-g k\right]+\sigma k^{3} a t z=0 \tag{2.2.7}
\end{equation*}
$$

We normalize the problem with the same procedure and get the following system: ODEs:

$$
\begin{gather*}
\left(U_{w}-c\right)\left(f_{w}^{\prime \prime}-k^{2} f_{w}\right)-U_{w}^{\prime \prime} f_{w}=0 \quad z \in[-\infty, 0]  \tag{2.2.8}\\
\left(U_{a}-c\right)\left(f_{a}^{\prime \prime}-k^{2} f_{a}\right)-U_{a}^{\prime \prime} f_{a}=0 \quad z \in[0, \infty] \tag{2.2.9}
\end{gather*}
$$

Boundary conditions:

$$
\begin{gather*}
f_{a}(0)=f_{w}(0)=c-U_{0}  \tag{2.2.10}\\
k f_{w}^{\prime}\left(c-U_{0}\right)+k f_{w} U_{w}^{\prime}-k F=\rho\left[k f_{a}^{\prime}\left(c-U_{0}\right)+k f_{a} U_{a}^{\prime}-k F\right]+W k^{3} \text { at } z=0  \tag{2.2.11}\\
f_{w}(z) e^{-k z}=\text { const, } z \rightarrow-\infty  \tag{2.2.12}\\
f_{a}(z) e^{k z}=\text { const, } z \rightarrow \infty \tag{2.2.13}
\end{gather*}
$$

### 2.3 The temporal case and the spatial case

The problem of wind wave generation describes the evolution of surface waves in time and space. When we being to deal with this problem we can ask questions such as: when does the wind start to blow? where does it blow from? how long is the fetch? is the wind time dependent? when does turbulence become dominant? and many more questions. Our model simplifies reality. The flow is two dimensional(2D), hence we do not discuss wind direction. The water depth and the air layer thickness are infinite.

We assume that the problem is steady. The flow in the model is time dependent but has a constant frequency, thus it is a quasi-steady state flow. In the problem we look for a combination of $\omega, k$ that will cause the solution of the ODEs to satisfy all of the boundary conditions. There are many combinations that we can find, but what is the meaning of every combination and which ones are important? First we should note that $\omega$ defines the frequency and growth rate in time and $k$ defines the wavelength and growth rate in space. Of course, from the physical aspect we are interested in the most unstable modes, because modes with large decay do not exist. In the open sea or even in the laboratory, waves will grow in time and space in a combination that is very hard to obtain. It should be more simple to think about waves that grow either in time only or in space only. These cases are possible in such a theoretical problem. The temporal case refers to growth in time only, as the spatial case refers to growth in space only. We can think of two scenarios which describe such cases. The first scenario is when the wind started to blow at a specific time $t=t_{0}$ in all of the space $-\infty<x<\infty$; in such a case there is no reason for waves to grow in space, and the $x$ dependence is fixed by the wave length only and all waves will grow in time only (temporal case). The second scenario is when the wind stared to blow from a specific location $x=x_{0}$ for all time values $-\infty<t<\infty$; in this case the waves start to develop in $x_{0}$ and they can grow only in space (spatial case). These two scenarios are theoretical scenarios, but when trying to measure wave growth in the laboratory these two cases are the most practical ones. When measuring temporal growth we measure the growth in the very initial time - since the moment when the wind started to blow. Since we want the amplitude to be small, this measuring will be at a fixed point $x=x_{0}$ for different time values and should be far from the edge. When measuring spatial growth, we measure the growth in a short length from the position that the wind started to blow. Since we want the amplitude to be small, this measuring will be at a fixed time $t=t_{0}$ for different positions. These two cases will be at the focus of this study.

### 2.4 Base flow profile

In the formulation, $U(z)$ represents the base flow. The base flow is sometimes called "background velocity" or "mean velocity profile". The base flow usually originates from the problem of flow without perturbations. For example, if the problem is on a
two-dimensional channel, the natural choice would be a parabolic profile which satisfies the laminar problem. In the problem of wave generation it is not so clear what the right choice is. If we look at the problem without waves - just air flow above water, we need to define the mean flow in each medium. The model as we write it is laminar; the idea is to insert the turbulence via the base flow. In the air medium, it is common to use the problem of turbulent air flow above a rough plate, which is sometimes called turbulent boundary layer. The solution of the turbulent problem contains fluctuating components and mean velocity. We will use the mean velocity profile as the base flow. The mean velocity profile has a logarithmic shape if we look far enough from the interface. The mean velocity profile is a solution of the boundary layer equation.

$$
\begin{equation*}
\underbrace{\mu \frac{d \bar{U}}{d z}}_{\text {laminar stress }}+\underbrace{\rho \kappa^{2} z^{2}\left(\frac{d \bar{U}}{d z}\right)^{2}}_{\text {turbulente stress }}=\tau_{0} \tag{2.4.1}
\end{equation*}
$$



Figure 2.1: The base flow and its first two derivatives for the numerical solution of the boundary layer equation with $U_{a}(0)=\frac{u_{*}}{2}$ for various values of $u_{*}$.

$$
\text { פרופיל הזרימה ונגזרותיו עבור פתרון נומרי של משוואת שכ } U_{a}(0)=\frac{u_{*}}{2} \text { ועבור ערכ הגבול הטורבולנטית, }
$$

This equation is based on Prandtl's mixing length theory under the main assumption in which the maximum length of an eddy depends on its distance from the wall. On the right side of the equation we write $\tau_{0}=\rho_{a} u_{*}^{2}$ - which is a constant, thus we
assume that the overall stress is constant with the vertical coordinate. This assumption can be problematic in the case of air water interface and transport of momentum between the air, the water and the waves. We will use an approximation for the boundary layer equation. The solution of the boundary layer equation is commonly divided into three regions: the first is around $z=0$ where the dominant term is the laminar stress, and the solution is approximately a linear velocity profile (laminar sub-layer). The third region $z \gg 1$ where the dominant term is the turbulent stress, and the solution is approximately logarithmic. Between the first and third region there is a buffer region; in this region we can not present an approximate analytic solution, but we can solve it numerically.

There are two main approximate profiles which were commonly used in previous studies. For Rayleigh's equation the common profile is:

$$
\begin{gather*}
U_{w} \equiv 0  \tag{2.4.2}\\
U_{a}=\frac{u_{*}}{\kappa} \log \left(1+\frac{z}{z_{0}}\right)  \tag{2.4.3}\\
z_{0}=\frac{\alpha_{c h} u_{*}^{2}}{g} ; \quad \alpha_{c h} \approx 0.014 ; \quad \kappa=0.41
\end{gather*}
$$

This profile is only an approximation and is characterized by a logarithmic shape for all values of $z$. The purpose of the " 1 " in the log argument is to make sure that $U_{a}(0)=0$. The relation for the roughness $z_{0}$ is Charnock's formula [7] and it is based on measurements, where $u_{*}$ is the friction velocity. This form of presentation is very convenient because there is only one parameter which defines the entire profile. In that case the first and second derivatives have the form:

$$
\begin{equation*}
U_{a}^{\prime}=\frac{u_{*}}{\kappa} \frac{1}{z+z_{0}} \tag{2.4.5}
\end{equation*}
$$

$$
\begin{equation*}
U_{a}^{\prime \prime}=-\frac{u_{*}}{\kappa} \frac{1}{\left(z+z_{0}\right)^{2}} \tag{2.4.6}
\end{equation*}
$$



Figure 2.2: The base flow and its first two derivatives of the logarithmic "one" profile for various values of $u_{*}$.

```
|u* פרופיל הזרימה ונגזרותיו עבור פרופיל לוגריתמי עם "1" ועבור ערכים שונים ש
```

As we can see from (2.4.5),(2.4.6) and Fig.2.2, higher $u_{*}$ means higher wind intensity, but when we look at the derivatives we see that a smaller value of $u_{*}$ causes larger derivatives at the interface.

For the viscous problem we will use the following base flow profile:
In the air (wind profile):

$$
U_{a}=\left\{\begin{array}{l}
U_{0}+\frac{u_{*}^{2}}{\nu_{a}} z \quad z \leq z_{1}  \tag{2.4.7}\\
U_{0}+m u_{*}+\frac{u_{*}}{\kappa}\left[\alpha-\tanh \left(\frac{\alpha}{2}\right)\right] \quad z \geq z_{1}
\end{array}\right.
$$

And in the water (current profile):

$$
\begin{equation*}
U_{w}=U_{0} \exp \left(\frac{\rho_{a} u_{*}^{2}}{U_{0} \mu_{w}} z\right) \tag{2.4.8}
\end{equation*}
$$

Where:

$$
\begin{equation*}
\alpha=\sinh ^{-1}(\beta), \quad \beta=\frac{2 \kappa u_{*}}{\nu_{a}}\left(z-z_{1}\right), \quad z_{1}=\frac{m \nu_{a}}{u_{*}}, \quad U_{0}=\frac{u_{*}}{2}, \quad m=5-8 \tag{2.4.9}
\end{equation*}
$$

This profile is also an approximation of the boundary layer equation. The profile has a linear segment with a possible offset and a segment that is asymptotically logarithmic. The parameter " $m$ " defines the thickness of the laminar sub layer (the linear segment)
and thus influences the derivative at the interface as well. This profile enables continuity of the function and its first derivative at the matching point $z=z_{1}$ for all value of the parameters " m " and $U_{0}$. The value of " m " is commonly set to 5 , see [11]. This base flow also has the quality of continuity of the shear stress between the air and the water. The first two derivatives are:

$$
\begin{gather*}
U_{a}^{\prime}=\left\{\begin{array}{l}
\frac{u_{*}^{2}}{\nu_{a}} \quad z \leq z_{1} \\
\frac{2 u_{*}^{2}}{\nu_{a} \sqrt{\beta^{2}+1}}\left[1-\frac{1}{2 \cosh ^{2}\left(\frac{\alpha}{2}\right)}\right] \quad z \geq z_{1}
\end{array}\right.  \tag{2.4.10}\\
U_{a}^{\prime \prime}=\left\{\begin{array}{l}
0 \quad z \leq z_{1} \\
\frac{4 u_{*}^{3} \kappa}{\nu_{a}^{2}\left(\beta^{2}+1\right)}\left[\frac{\beta}{\sqrt{\beta^{2}+1}}\left(\frac{1}{2 \cosh ^{2}\left(\frac{\alpha}{2}\right)}-1\right)+\frac{\sinh \left(\frac{\alpha}{2}\right)}{2 \cosh ^{3}\left(\frac{\alpha}{2}\right)}\right] \quad z \geq z_{1}
\end{array}\right. \tag{2.4.11}
\end{gather*}
$$



Figure 2.3: The base flow and its first two derivatives of the linear-logarithmic profile

$$
\text { for various values of } u_{*}(m=5) .
$$

פרופיל הזרימה ונגזרותיו עבור פרופיל ליניארי־לוגריתמי כאשר $U_{a}(0)=\frac{u_{*}}{2}$ ועבור ערכים

$$
\text { שונים של } m=5) u_{*}
$$

As can be seen from (2.4.10),(2.4.11) and Fig.2.3, the parameter $u_{*}$ defines the wind intensity and also defines the current-profile. The first derivative has a discontinuity
at the interface, but is continues in the air and has a greater value when $u_{*}$ is larger. The second derivative also has discontinuity at the interface, and has a greater absolute value when $u_{*}$ is larger. The difference between the "one" profile and the lin-log profile is: in the lin-log profile we can set the drift current, while in the "one" profile there is no current at all; in the lin-log profile the second derivative at the interface is zero, while in the "one" profile it is very big; in the lin-log profile the first derivative at the interface becomes higher when increasing $u_{*}$, while in the "one" profile it becomes higher when $u_{*}$ decreases. Note that $u_{*}$ is the friction velocity and it is an indicator for the wind intensity, but for the same $u_{*}$ we can get different values of $U_{a}$ for different profiles, see Fig.2.4. As we can see from Fig.2.1,2.3,2.2 and 2.4, the behavior of the lin-log profile is much more similar to the numerical solution of the boundary layer equation; for most values of $u_{*}$ and $z$ the wind speed of the numerical solution gets in between the "one" and the lin-log profile, where commonly the lin-log is overestimated and the "one" is underestimated.


Figure 2.4: Comparison of wind intensity at $z=10[m]$ and $z=0.1[m]$ for the numerical solution,"one" and lin-log profiles.

השוואת עוצמת הרוח בגובה 10מ׳ ובגובה 10ס״מ עבור פרופילי מהירות שונים

## Chapter 3

## Numerical Methods

### 3.1 Numerical methods for the inviscid model, Rayleigh's equation

Rayleigh's equation is a second order differential equation. The problem is an eigenvalue problem which is defined as:

$$
\begin{gather*}
\left(U_{w}-c\right)\left(f_{w}^{\prime \prime}-k^{2} f_{w}\right)-U_{w}^{\prime \prime} f_{w}=0 \quad z \in[-\infty, 0]  \tag{3.1.1}\\
\left(U_{a}-c\right)\left(f_{a}^{\prime \prime}-k^{2} f_{a}\right)-U_{a}^{\prime \prime} f_{a}=0 \quad z \in[0, \infty] \tag{3.1.2}
\end{gather*}
$$

Boundary conditions:

$$
\begin{gather*}
f_{a}(0)=f_{w}(0)=c-U_{0}  \tag{3.1.3}\\
k f_{w}^{\prime}\left(c-U_{0}\right)+k f_{w} U_{w}^{\prime}-F=\rho\left[k f_{a}^{\prime}\left(c-U_{0}\right)+k f_{a} U_{a}^{\prime}-F\right]+W k^{3} \text { at } z=0  \tag{3.1.4}\\
f_{w}(z) e^{-k z}=\mathrm{const}, z \rightarrow-\infty  \tag{3.1.5}\\
f_{a}(z) e^{k z}=\mathrm{const}, z \rightarrow \infty \tag{3.1.6}
\end{gather*}
$$

Since the problem is linear, we can take (3.1.3) as one condition $f_{a}(0)=f_{w}(0)$ and hence have four homogenous boundary conditions for the two coupled equations. We chose a different method of solution which produces only one eigenvalue in each search process. The reasons for this choice are: first we are interested in a very narrow range in the eigenvalue plane, where the eigenvalues have an interesting physical meaning. Secondly, due to numerical aspects, searching for all of the eigenvalues for such a problem would be a very expensive calculation. The process which is used is such, in which
we have coupled second order equations with five boundary conditions. The process of searching the eigenvalue is: choosing an initial guess for the eigenvalue, solving the boundary value problem with four boundary conditions, two at $z \rightarrow \pm \infty$ and two at the interface $z=0$, then using the extra condition (dynamic boundary condition) in order to get an improved guess. This iterative process uses the dynamic boundary condition (3.1.4) as the extra condition. Since the differential equations are linear, we can simply apply a shooting method in order to solve them. The shooting method is a method which is used to solve a boundary value problem with the use of the solution of an initial value problem. Initial value problems are much simpler to solve because the solution at every point depends on the previous point only. Hence, it is possible to use high order schemes without the need of solving large systems of equations. The advancement in the solution is sometimes called integration. Another benefit is that one can integrate the equation in every direction. Generally, these linear equations have two independent solutions. Asymptotically, for $z \rightarrow \pm \infty$ when $U_{a}^{\prime \prime} \rightarrow_{z \rightarrow \pm \infty} 0$ we can obtain that the independent solutions for both equations are $e^{k z}, e^{-k z}$. Hence, it is clear that there is a pair of independent solutions, one of them having an exponential decay and the other an exponential blow-up. The blowing-up solution, for example $e^{k z}$ in the air, has a relation between the function and its derivative of $f_{a}^{\prime}(\infty)=k f_{a}(\infty)$, whereas the decaying solution has the relation of $f_{a}^{\prime}(\infty)=-k f_{a}(\infty)$. Since we are interested in a decaying solution in both media, it will be clever to start the integration from $\pm \infty$ with the following initial conditions:

$$
\begin{align*}
& f_{a}=1, f_{a}^{\prime}=-k, \quad z \rightarrow \infty  \tag{3.1.7}\\
& f_{w}=1, f_{w}^{\prime}=k, \quad z \rightarrow-\infty
\end{align*}
$$

These initial conditions have the same meaning as conditions (3.1.5),(3.1.6). After integrating both equations, in the water from negative infinity to zero and in the air from infinity to zero, we simply have to normalize the solution with the right factor so that it will fit condition (3.1.3). After obtaining the solution, we can calculate the two unknown values for the dynamic boundary condition $f_{a}^{\prime}(0), f_{w}^{\prime}(0)$, and then calculate the improved guess. For the integration of the equations we use a MATLAB initial value solver ode45. The ode45 solver is based on an explicit Runge-Kutta $(4,5)$ formula, the Dormand-Prince pair. It is a one-step solver - in computing, $y\left(t_{n}\right)$, it needs only the solution at the immediate preceding time point, $y\left(t_{n-1}\right)$, and it is a variable step solver.

In the calculation infinity will be regarded as a finite value $z_{\infty}$. This value needs to be relative to the wave length, since long waves are influenced by a thicker air/water layer. Since the $z$ coordinate is normalized by the wave-number, we set $z_{\infty}$ and can test the sensitivity to this value.

### 3.2 Numerical methods for the viscous model, Orr-Sommerfeld equation

In the viscous model, the governing equation is the Orr-Sommerfeld equation. This problem is also an eigenvalue problem. Since the governing equation is of fourth order, there are more boundary conditions than in the previous problem. Applying a shooting method to this problem is not so simple, because the initial value solver has stability problems. Hence, we try to look for another numerical method which will fit this kind of problem. There are a few options we can choose from, for instance: adding filtering to the integration process, using finite differences, or using collocation methods. After applying a finite- differences method which evidently produced disappointing results, we chose the Chebyshev collocation method.

### 3.2.1 About the Chebyshev collocation method

In order to present the numerical solution by the Chebyshev collocation method, we first have to explain the idea behind it, which is based on spectral methods. The mathematical idea is to use a series expansion to represent the unknown function. In such an expansion, we need to define the basis functions; these functions need to be orthogonal and usually orthonormal, for example sin, cos. These functions are the eigenfunctions of specific Sturm-Liouville problems. Since these functions are eigenfunctions, we can present every function as an infinite series expansion. In our case we will use Chebyshev polynomials as the eigenfunctions. These polynomials are defined on the interval $x \in[-1,1]$ and their values are also within $T_{n}(x) \in[-1,1]$. The explicit expression for the $n t h$ Chebyshev polynomial is:

$$
\begin{equation*}
T_{n}(x)=\frac{n}{2} \sum_{m=0}^{[n / 2]}(-1)^{m} \frac{(n-m-1)!}{m!(n-2 m)!}(2 x)^{n-2 m} \tag{3.2.1}
\end{equation*}
$$



Figure 3.1: First ten Chebyshev polynomials עשרת פולינומי צ'בישב הראשונים

There are many recursion relations for the Chebyshev polynomials and their derivatives. We first calculate the first and second polynomials and then obtain the others by recursion relations. For example we can use:

$$
\begin{equation*}
T_{n+1}(x)=2 x T_{n}(x)-T_{n-1}(x) \tag{3.2.2}
\end{equation*}
$$

And then the first 4 polynomials are:

$$
\begin{equation*}
T_{0}=1, T_{1}=x, T_{2}=2 x^{2}-1, T_{3}=4 x^{3}-3 x \tag{3.2.3}
\end{equation*}
$$

As already mentioned, we shall represent the unknown function using the series of Chebyshev polynomials.

$$
\begin{equation*}
f(x)=\sum_{n=0}^{\infty} a_{n} T_{n}(x) \tag{3.2.4}
\end{equation*}
$$

Since we are interested in a numerical approximation, we will use a finite Series. The essence of finding the solution is to find the coefficients of the series $a_{n}$. For example: for a given function $f(x)$ and given grid points $\left(x_{1}, x_{2},,, x_{n}\right)$, we can obtain the coefficients by a simple algebraic operation. For every grid point the following equation can be written:

$$
\begin{equation*}
a_{0} T_{0}\left(x_{i}\right)+a_{1} T_{1}\left(x_{i}\right)+a_{2} T_{2}\left(x_{i}\right)+\ldots+a_{N} T_{N}\left(x_{i}\right)=f\left(x_{i}\right) \forall i=1,2,,, N \tag{3.2.5}
\end{equation*}
$$

Now we can define the matrix $T$ to be:

$$
T=\left[\begin{array}{ccccc}
T_{0}\left(x_{1}\right) & T_{1}\left(x_{1}\right) & \cdot & \cdot & T_{N}\left(x_{1}\right)  \tag{3.2.6}\\
T_{0}\left(x_{2}\right) & \cdot & & & \\
\cdot & & \cdot & & \\
\cdot & & & \cdot & \\
T_{0}\left(x_{N}\right) & T_{1}\left(x_{N}\right) & \cdot & . & T_{N}\left(x_{N}\right)
\end{array}\right]
$$

Thus the linear system for the coefficients is (where $\vec{a}=\left(a_{1}, a_{2},,, a_{N}\right)$ ):

$$
\begin{equation*}
T \vec{a}=\vec{f} \Rightarrow \vec{a}=T^{-1} \vec{f} \tag{3.2.7}
\end{equation*}
$$

When using this method to solve differential equations, $f$ will play the role of the unknown function. In order to represent the derivatives of $f$ we can write:

$$
\begin{equation*}
f^{(n)}=T^{(n)} \vec{a} \Rightarrow f^{(n)}=T^{(n)} T^{-1} \vec{f}=D_{n} \vec{f} \tag{3.2.8}
\end{equation*}
$$

As we can understand from the above equation, it is easier and more practical to use $\vec{f}$ than to use $\vec{a}$, because $\vec{a}$ is unknown. From the above equation we define the differentiation matrix $D_{n}$ to be:

$$
\begin{equation*}
D_{n}=T^{(n)} T^{-1} \tag{3.2.9}
\end{equation*}
$$

These matrices depend on the grid points. The grid which we will use for the Chebyshev polynomials is:

$$
\begin{equation*}
x_{j}=\cos \left(\frac{j \pi}{N}\right) \forall j=0,1,, N \tag{3.2.10}
\end{equation*}
$$

If we want to use it to solve a linear ODE, we simply need to transfer the differential equation into a linear system by substituting the above relation. For example: in order to solve the simple BVP(boundary value problem):

$$
\begin{equation*}
y^{\prime \prime}+y=0 ; y(0)=1, y(1)=2 \tag{3.2.11}
\end{equation*}
$$

We transfer the equation to:

$$
\begin{equation*}
D_{2} \vec{f}+\vec{f}=\left(D_{2}+I\right) \vec{f}=0 \tag{3.2.12}
\end{equation*}
$$

In order to satisfy the boundary condition, we exchange the first and last rows with the first and last rows of the identity matrix, respectively; and the first and last elements in the right hand side vector to be 1 and 2 , respectively. Then we need only solve the linear system to get the solution.

### 3.2.2 Applying the method to the problem

The mathematical problem:
Governing equation:
The ODEs:

$$
\begin{align*}
& i R_{w}^{-1}\left(f_{w}^{(4)}-2 k^{2} f_{w}^{\prime \prime}+k^{4} f_{w}\right)+k\left[\left(U_{w}-c\right)\left(f_{w}^{\prime \prime}-k^{2} f_{w}\right)-U_{w}^{\prime \prime} f_{w}\right]=0 \quad z \in(-\infty, 0]  \tag{3.2.13}\\
& i R_{a}^{-1}\left(f_{a}^{(4)}-2 k^{2} f_{a}^{\prime \prime}+k^{4} f_{a}\right)+k\left[\left(U_{a}-c\right)\left(f_{a}^{\prime \prime}-k^{2} f_{a}\right)-U_{a}^{\prime \prime} f_{a}\right]=0 \quad z \in[0, \infty) \tag{3.2.14}
\end{align*}
$$

Boundary conditions:

$$
\begin{gather*}
f_{a}(0)=f_{w}(0)=c-U_{0}  \tag{3.2.15}\\
f_{w}^{\prime}+U_{w}^{\prime}=f_{a}^{\prime}+U_{a}^{\prime} \text { at } z=0  \tag{3.2.16}\\
\mu\left(f_{a}^{\prime \prime}+k^{2} f_{a}+U_{a}^{\prime \prime}\right)=\left(f_{w}^{\prime \prime}+k^{2} f_{w}+U_{w}^{\prime \prime}\right) \text { at } z=0  \tag{3.2.17}\\
k f_{w}^{\prime}\left(c-U_{0}\right)+k f_{w} U_{w}^{\prime}+i R_{w}^{-1}\left(3 k^{2} f_{w}^{\prime}-f_{w}^{\prime \prime \prime}\right)-F= \\
=\rho\left[k f_{a}^{\prime}\left(c-U_{0}\right)+k f_{a} U_{a}^{\prime}+i R_{a}^{-1}\left(3 k^{2} f_{a}^{\prime}-f_{a}^{\prime \prime \prime}\right)-F\right]+W k^{3} \text { at } z=0  \tag{3.2.18}\\
f_{w}(z) e^{-k z}=\text { const, } z \rightarrow-\infty  \tag{3.2.19}\\
f_{a}(z) e^{k z}=\text { const, } z \rightarrow \infty \tag{3.2.20}
\end{gather*}
$$

In this problem we need to find the complex frequency, or the complex wavenumber, which causes the ODE to satisfy the boundary conditions. Since we can rearrange the problem as a homogeneous system, it is an eigenvalue problem. It can be divided into two special cases: the case of temporal growth and the case of spatial growth. For the case of temporal growth it is a linear eigenvalue problem, but for the case of spatial growth it is a nonlinear eigenvalue problem. For numerical reasons, we prefer to reduce the problem in each medium into two dependent second order equations, in the following manner. Instead of the governing equation we have the set:

$$
\begin{align*}
& V-f^{\prime \prime}=0  \tag{3.2.21}\\
& i R^{-1}\left(V^{\prime \prime}-2 k^{2} V+k^{4} f_{a}\right)+k\left[\left(U_{a}-c\right)\left(V-k^{2} f_{a}\right)-U_{a}^{\prime \prime} f_{a}\right]=0
\end{align*}
$$

Where $V \triangleq f^{\prime \prime}$. The ODE system includes equations for the water and the air media, and it is linear in the unknown function $f(z)$. It was solved in two ways: the first is to build a linear algebraic generalized eigenvalue problem $(A v=\lambda B v)$ and to find all of the eigenvalues and eigenfunctions using a MATLAB solver (only for the temporal
case); the second way is to build the solution by an iterative process which converges to a specific frequency/wavenumber which satisfies the system. In this text we will focus on the iterative process. The iterative process contains the following stages: 1.Guessing a value for $\omega, k$. 2 .Solving the boundary value problem, which is inhomogeneous due to the kinematic boundary condition. 3.Substituting the solution into the dynamic boundary condition. 4.Calculating an improved guess for $\omega, k$, using a standard root search method. The process converges to a specific value of $\omega, k$ which is one solution of the problem. The value which the process converges to depends on the initial guess. Different initial guesses can lead to a different eigenvalues. In order to achieve good accuracy it is imperative to work with large matrices. Since using a generic eigenvalue MATLAB solver causes significant errors in such large matrices its use is found to be inefficient. Another reason not to use the eigenvalue solver, is that we are not interested in all of the eigenvalues, we are interested only in those which have an important physical interpretation. Another improvement to the regular method is to divide the air domain into two different domains. This trick (based on [3]) can help improve the calculation efficiency in problems with sharp boundary layer - which have a great influence on the solution. The first region will be very close to the interface and as a result it requires a very fine grid. The thickness of the first region is the same as the linear segment in the lin-log profile. The second and major segment will be from that point and on. This division divides the air medium into two artificial layers. Between these two layers we also need to satisfy the boundary conditions which are similar to those in the air-water interface.

When solving the problem numerically, we need to approximate the infinity by a finite value. Choosing that value is naturally related to the wave length. The value which we commonly use is $z_{\infty}=10$ which means that $z_{\infty \text { dimensional }} \approx 1.6 \lambda$, the sensitivity for that value is tested in the chapter about validation of the numerical results. Since the natural domain of Chebyshev's polynomials is $[-1,1]$ and we use different domains, we need to use transformations. We will use linear transformations which are the most simple way and do not change the equations. The transformations are:

$$
\begin{equation*}
z_{w}=\frac{(x-1) z_{\infty}}{2}, \quad z_{a 1}=\frac{(x+1) z_{1}}{2}, \quad z_{a 2}=\frac{(x+1)\left(z_{\infty}-z_{1}\right)}{2}+z_{1} \tag{3.2.22}
\end{equation*}
$$

Where $x \in[-1,1]$ is the Chebyshev coordinate and $z_{w}, z_{a 1}, z_{a 2}$ are the $z$ coordinate in each part of the physical domain. When using these transformations we need to use


## Water



Figure 3.2: Computational domain of the problem התחום החישובי של הבעיה
the chain rule; since the transformation is linear it is simply $\frac{d^{n}}{d z^{n}}=\left(\frac{d x}{d z}\right)^{n} \frac{d^{n}}{d x^{n}}$. When applying the method, we build a linear system for the collocation points. In order to do that, we need to define the order of the unknown points vector as:

$$
\begin{equation*}
\vec{F}=\left[\vec{V}_{a 2}^{t}, \vec{f}_{a 2}^{t}, \vec{V}_{a 1}^{t}, \vec{f}_{a 1}^{t}, \vec{V}_{w}^{t}, \vec{f}_{w}^{t}\right]^{t} \tag{3.2.23}
\end{equation*}
$$

Where each one of these vectors is a column vector and the upper term in each vector is corresponding to the upper point in the physical grid. For example, the first term in $\vec{V}_{a 2}$ corresponds to the point $z=z_{\infty}$ and the last term in $\vec{f}_{a 1}$ corresponds to the point $z=0$. Each pair of these vectors can be in a different length because we can control the number of grid points in each medium. The next stage is to transfer the differential equations into an algebraic equation with the use of the Chebyshev differentiation matrices in the form (where $D_{n}=\left(\frac{d x}{d z}\right)^{n} \hat{D}_{n}$ and $\hat{D}$ is the original differentiation matrix).

$$
\begin{align*}
& I V-D_{2} f=0  \tag{3.2.24}\\
& \left(D_{2}-\operatorname{diag}\left(2 k^{2}-i k R(c-U)\right)\right) V-\operatorname{diag}\left(i k R\left(k^{2}(c-U)-U^{\prime \prime}\right)\right) f=0
\end{align*}
$$

This set represents the equations for each fluid layer and it is a block matrix.

$$
\left[\begin{array}{lc}
{[I]} & {\left[-D_{2}\right]}  \tag{3.2.25}\\
{\left[D_{2}-\operatorname{diag}\left(2 k^{2}-i k R(c-U)\right)\right]\left[-\operatorname{diag}\left(i k R\left(k^{2}(c-U)-U^{\prime \prime}\right)\right)\right]}
\end{array}\right]\left[\begin{array}{c}
\vec{V} \\
\vec{f}
\end{array}\right]=0
$$

After building this block matrix for each layer, we build one big block matrix for the whole problem.

As it seems in the above equation, the system is homogenous but it is not the whole set, because we did not apply the boundary conditions yet. Applying the boundary conditions in this collocation method is similar to the application in the finite-differences method; we simply replace several equations in the system with the approximation of the boundary condition. It can be very important which equation will be replaced and how to do so. First we will approximate the boundary conditions at infinity by:

$$
\begin{equation*}
f_{a}\left(z_{\infty}\right)=0, V_{a}\left(z_{\infty}\right)=0, f_{w}\left(-z_{\infty}\right)=0, V_{w}\left(-z_{\infty}\right)=0 \tag{3.2.27}
\end{equation*}
$$

These boundary conditions can be applied by simply erasing the appropriate row and column before solving the system, which are with the following indices: $1, N_{a 2}+$ $1,2 N_{a 2}+2 N_{a 1}+N_{w}, 2 N_{a 2}+2 N_{a 1}+2 N_{w}$. Now, applying the interface boundary conditions. We have two interfaces: the first is the real interface between the $A_{1}$ layer and the water, and the second is the artificial interface between $A_{2}$ and $A_{1}$. At the air-water interface we need to apply (3.2.15), which is two conditions: (3.2.16), and (3.2.17). Each one of these four conditions replaces the equation in one of the collocation points. The equations are:

$$
\begin{gather*}
{\left[[\text { zeros }]_{1 \times 2 N_{a 2}+N_{a 1}}, I_{a 1,(\text { last row })},[\text { zeros }]_{1 \times 2 N_{w}}\right] \vec{F}=c-U_{0}}  \tag{3.2.28}\\
{\left[[\text { zeros }]_{1 \times 2 N_{a 2}+2 N_{a 1}+N_{w}}, I_{w,(\text { first row })}\right] \vec{F}=c-U_{0}}  \tag{3.2.29}\\
{\left[[\text { zeros }]_{1 \times 2 N_{a 2}+N_{a 1}}, D_{a 1,(\text { last row }),},[\text { zeros }]_{1 \times N_{w}},\right.}  \tag{3.2.30}\\
\left.,-D_{w,(\text { first row })}\right] \vec{F}=U_{w}^{\prime}(0)-U_{a}^{\prime}(0) \\
{\left[[\text { zeros }]_{1 \times 2 N_{a 2},}, \mu I_{a 1,(\text { last row })}, \mu k^{2} I_{a 1,(\text { last row })},\right.}  \tag{3.2.31}\\
\left.,-I_{w,(\text { first row })},-k^{2} I_{w,(\text { first row })}\right] \vec{F}=U_{w}^{\prime \prime}(0)-\mu U_{a}^{\prime \prime}(0)
\end{gather*}
$$

The rows in the matrix which will be replaced have the following indices: $2 \mathrm{Na} 2+$ $N_{a 1}, 2 N a 2+2 N_{a 1}, 2 N a 2+2 N_{a 1}+1,2 N a 2+2 N_{a 1}+N_{w}$. It does not matter which condition replaces which collocation point. In the air1-air2 interface the boundary conditions are similar to the air-water interface, except that the condition (3.2.15) now gives only one condition and therefore we have to use the dynamic boundary condition (3.2.18). The conditions get a much simpler form because the density and viscosity ratio are equal 1. Another simplification is the air profile used which has a continues first derivative. Hence, the first two conditions will be a continuity of the function and its first derivative, and the dynamic boundary condition has the meaning of continuity of the third derivative. The collocation points which will be replaced are: $N_{a 2}, 2 N_{a 2}, 2 N_{a 2}+1,2 N_{a 2}+N_{a 1}$. And the equations take the form:

$$
\begin{align*}
& f_{a 2}(h)=f_{a 1}(h) \\
& f_{a 2}^{\prime}(h)=f_{a 1}^{\prime}(h) \\
& f_{a 2}^{\prime \prime \prime}+U_{a 2}^{\prime \prime}=f_{a 1}^{\prime \prime}+U_{a 1}^{\prime \prime} \text { at } z=h \\
& f_{a 2}^{\prime \prime \prime}=f_{a 1}^{\prime \prime \prime} \text { at } z=h \\
& {\left[[z e r o s]_{1 \times N_{a 2}}, I_{a 2,(\text { last row) }}, I_{a 1,(\text { first row })},[\text { zeros }]_{1 \times N_{a 1}+2 N_{w}}\right] \vec{F}=0} \\
& {\left[[z e r o s]_{1 \times N_{a 2}}, D_{a 2,(\text { last row })},-D_{a 1,(\text { first row })},[z e r o s]_{1 \times N_{a 1}+2 N_{w}}\right] \vec{F}=0} \\
& {\left[I_{a 2,(\text { last row })},[\text { zeros }]_{1 \times N_{a 2}},-I_{a 1,(\text { first row })},[z e r o s]_{1 \times N_{a 1}+2 N_{w}}\right] \vec{F}=U_{a 1}^{\prime \prime}(h)-U_{a 2}^{\prime \prime}(h)}  \tag{3.2.38}\\
& {\left[D_{a 2,(\text { last row })},[z e r o s]_{1 \times N_{a 2}},-D_{a 1,(\text { first row) }},[z e r o s]_{1 \times N_{a 1}+2 N_{w}}\right] \vec{F}=0} \tag{3.2.39}
\end{align*}
$$

After applying all of the boundary conditions, first the interface boundary conditions by replacing the right equation with the right boundary condition equation and then the infinity boundary conditions by erasing the right rows and columns, we solve the linear system by using a standard MATLAB solver. This stage is called the BVP solver and the output is simply the functions $f(z), V(z)$. The only condition which is not used yet is the dynamic boundary condition at the air-water interface. We will use this
condition in order to build an iterative process which converges to the eigenvalue. We can define:

$$
\begin{array}{r}
G(k, \omega)=k f_{w}^{\prime}\left(c-U_{0}\right)+k f_{w} U_{w}^{\prime}+i R_{w}^{-1}\left(3 k^{2} f_{w}^{\prime}-f_{w}^{\prime \prime \prime}\right)-F- \\
-\rho\left[k f_{a}^{\prime}\left(c-U_{0}\right)+k f_{a} U_{a}^{\prime}+i R_{a}^{-1}\left(3 k^{2} f_{a}^{\prime}-f_{a}^{\prime \prime \prime}\right)-F\right]-W k^{3}=0 \text { at } z=0 \tag{3.2.40}
\end{array}
$$

This is simply another form for writing the dynamic boundary condition. Note that after we solve the boundary value problem we know all of the quantities in the equation. It is clear that we want such $\omega, k$ which causes $G(k, \omega)=0$. The iterative process we use is based on a secant method for the problem $G(k, \omega)=0$. Note that we only look for one of the quantities $\omega, k$ because we deal with either the temporal or the spatial case. The secant method is simply:

$$
\begin{equation*}
\omega_{n+1}, k_{n+1}=\omega_{n}, k_{n}+\frac{G\left(\omega_{n}, k_{n}\right)\left(\omega_{n}, k_{n}-\omega_{n-1}, k_{n-1}\right)}{G\left(\omega_{n}, k_{n}\right)-G\left(\omega_{n-1}, k_{n-1}\right)} \tag{3.2.41}
\end{equation*}
$$

The choice of the secant method is because of its simplicity and the fact that there is no need to calculate the derivative or any other additional calculation. Note that $G$ is a complex function in complex variable $\omega, k$.


Figure 3.3: Orr-Sommerfeld solver flow chart

## Chapter 4

## Validation of the Numerical

## Results

In this chapter we will try to prove that the numerical results are in fact close to the true results of the problem. Another target is to show that the approximations which we had made are valid. The tools which we will use are: first to show that the results are not sensitive to the numerical parameters (convergence), second to compare the results with analytical calculations (test case)and third to compare the results with previous studies.

### 4.1 Validation of the inviscid case

As described in the previous chapter, we use MATLAB initial value solver in order to integrate Rayleigh's equation. Since it is a varying step solver it adapts the step size to its location.

### 4.1.1 Sensitivity to $z_{\infty}$

Since we practically need to use a finite interval for the numerical calculations, we have to show that our choice does not have a major effect on the results. The $z$ coordinate is normalized by $k_{0}$ in the form $z=z_{\text {dimensional }} k_{0}$. From the physical aspect, the value of $z_{\infty}$ should be related to the wave length. From the linear theory of water waves, it is known that after half of a wavelength the influence of the waves on the flow field is

|  | $\lambda=0.1 m, \quad u_{*}=0.3 \mathrm{~m} / \mathrm{sec}$ | $\lambda=0.1 \mathrm{~m}, \quad u_{*}=1 \mathrm{~m} / \mathrm{sec}$ |
| :--- | :--- | :--- |
| $z_{\infty}=2$ | $0.98175865311762+0.00807019480728 \mathrm{i}$ | $1.02387462484237+0.05182653075216 \mathrm{i}$ |
| $z_{\infty}=10$ | $0.98176172076009+0.00806665342400 \mathrm{i}$ | $1.02386080291654+0.05184138532963 \mathrm{i}$ |
| $z_{\infty}=100$ | $0.98176172076010+0.00806665342398 \mathrm{i}$ | $1.02386080291647+0.05184138532970 \mathrm{i}$ |

Table 4.1: Sensitivity to $z_{\infty}$ (values of $\omega$ ), temporal case, "one" profile רגישות ל־ס
minor. The value which we use for most of our calculations is $z_{\infty}=10$, which means that $z_{\infty, \text { dimensional }}=\frac{10}{2 \pi} \lambda=1.591 \lambda$. In order to justify this value, we made a few runs with different values for $z_{\infty}$. See Table (4.1).

### 4.1.2 Comparison with previous studies

The next level of validation is to compare the numerical results with those who made similar calculations before. In Stiassnie et al. [21] Fig. 3 they compare their results with Komen et al. [12] and some experimental data. Their model neglects the surface tension and they use the "one" wind profile. See Fig. (4.1).


Figure 4.1: Normalized growth rate vs. normalized friction velocity temporal case.
Comparison with previous studies. (similar to Fig. 3 in [21]).
ערכי הגידול המנורמל כתלות במהירות החיכוך המנורמלת, השוואה עם עבודות קודמות, התפתחות בזמן

### 4.2 Validation of the viscous case

In this section we will show the convergence of the Chebyshev collocation method, test the sensitivity of the model to the interval size and compare the results with the test case and previous studies.

### 4.2.1 Analytical solution of the viscous problem for linear wind profile and constant current

In this section we present the analytical solution for the case of linear wind profile and constant current. The profile has the form:

$$
\begin{equation*}
U_{a}=U_{0}+a z, \quad U_{w}=U_{0} \tag{4.2.1}
\end{equation*}
$$

The Orr-Sommerfeld equation is:

$$
\begin{equation*}
i \epsilon\left(f^{(4)}-2 k^{2} f^{\prime \prime}+k^{4} f\right)+k\left[(U-c)\left(f^{\prime \prime}-k^{2} f\right)-U^{\prime \prime} f\right]=0 \tag{4.2.2}
\end{equation*}
$$

Where $\epsilon_{w, a}=\frac{1}{R e_{w, a}}=\frac{\nu_{w, a} k_{0}^{2}}{\omega_{0}}$ is the inverse Reynolds number. Since we deal with linear wind profile which has the form $U=U_{0}+a z$. Where $U_{0}$ is the drift velocity and "a" is the velocity slope. In such a profile $U^{\prime \prime} \equiv 0$. After substituting it into the equation it has the form:

$$
\begin{equation*}
i \epsilon\left(f^{(4)}-2 k^{2} f^{\prime \prime}+k^{4} f\right)+k(U-c)\left(f^{\prime \prime}-k^{2} f\right)=0 \tag{4.2.3}
\end{equation*}
$$

Or:

$$
\begin{equation*}
f^{(4)}-k f^{\prime \prime}\left(2 k+\frac{i}{\epsilon}(U-c)\right)+k^{3} f\left(k+\frac{i}{\epsilon}(U-c)\right)=0 \tag{4.2.4}
\end{equation*}
$$

Now we can define $F \triangleq f^{\prime \prime}-k^{2} f$. Hence the equation becomes:

$$
\begin{equation*}
F^{\prime \prime}-k\left(k+\frac{i}{\epsilon}(U-c)\right) F=0 \tag{4.2.5}
\end{equation*}
$$

By the transformation of variables, we can transform the equation into Airy's equation.

$$
\begin{equation*}
F^{\prime \prime}(u)-u F(u)=0 \tag{4.2.6}
\end{equation*}
$$

In order to find such a transformation, we look at a transformation of the form $u=H k\left(k+\frac{i}{\epsilon}(U-c)\right)$ (where $H$ is a constant) and substitute it into the equation using the chain rule.

$$
\begin{equation*}
-\left(H \frac{k a}{\epsilon}\right)^{2} F^{\prime \prime}-k\left(k+\frac{i}{\epsilon}(U-c)\right) F=0 \Rightarrow F^{\prime \prime}-\frac{k\left(k+\frac{i}{\epsilon}(U-c)\right)}{-\left(H \frac{k a}{\epsilon}\right)^{2}} F=0 \tag{4.2.7}
\end{equation*}
$$

Hence:

$$
\begin{equation*}
\frac{1}{-\left(H \frac{k a}{\epsilon}\right)^{2}}=H \Rightarrow H=\left(\frac{i \epsilon}{k a}\right)^{\frac{2}{3}} \tag{4.2.8}
\end{equation*}
$$

Where:

$$
\begin{equation*}
u=\left(\frac{i \epsilon}{k a}\right)^{\frac{2}{3}} k\left(k+\frac{i}{\epsilon}(U-c)\right)=\frac{k}{2}(1+i \sqrt{3})\left(\frac{\epsilon}{k a}\right)^{\frac{2}{3}}\left(k+\frac{i}{\epsilon}(U-c)\right) \tag{4.2.9}
\end{equation*}
$$

Thus, $F(u(z))$ is a solution of Airy's equation. Since it is a second order equation, it has two independent solutions. There are a few common pairs of independent solutions, see [1],[25].

$$
\begin{align*}
& A i(u), B i(u) \\
& A i(u), A i\left(u e^{\frac{2 \pi i}{3}}\right)  \tag{4.2.10}\\
& A i(u), A i\left(u e^{-\frac{2 \pi i}{3}}\right)
\end{align*}
$$

Since the solution and its derivative must vanish at infinity, We choose the pair $A i(u), A i\left(u e^{-2 / 3 \pi i}\right)$. Hence, the solution is:

$$
\begin{equation*}
F=f^{\prime \prime}-k^{2} f=c_{1} A i(u)+c_{2} A i\left(u e^{-\frac{2 \pi i}{3}}\right) \tag{4.2.11}
\end{equation*}
$$

At this stage, we need to look at the asymptotic behavior of the independent solutions for large values of $|z|$ in order to be sure that we satisfy the boundary condition at infinity. As mentioned in [25], the Airy function $A i$ blows up for large $|u|$ outside the section $|\arg (u)|<\frac{\pi}{3}$. Hence, we should look at the behavior of $\arg (u)$ at infinity $z \rightarrow$ $\pm \infty$.

$$
\begin{align*}
& \arg (u)= \begin{cases}\frac{5 \pi}{6}+\frac{\arg (k)}{3} & z \rightarrow \infty \\
-\frac{\pi}{6}+\frac{\arg (k)}{3} & z \rightarrow-\infty\end{cases}  \tag{4.2.12}\\
& \arg \left(u e^{\frac{-2 \pi i}{3}}\right)= \begin{cases}\frac{\pi}{6}+\frac{\arg (k)}{3} & z \rightarrow \infty \\
-\frac{5 \pi}{6}+\frac{\arg (k)}{3} & z \rightarrow-\infty\end{cases} \tag{4.2.13}
\end{align*}
$$

If we take $\Re(k)>0 \Rightarrow|\arg (k)|<\frac{\pi}{2}$ we can be sure that:

$$
\begin{align*}
|\arg (u)|<\frac{\pi}{3} & z \rightarrow-\infty  \tag{4.2.14}\\
\left|\arg \left(u e^{\frac{-2 \pi i}{3}}\right)\right|<\frac{\pi}{3} & z \rightarrow \infty \tag{4.2.15}
\end{align*}
$$

Thus in the air:

$$
\begin{equation*}
F_{a}=f_{a}^{\prime \prime}-k^{2} f_{a}=c_{1} A i\left(u e^{-\frac{2 \pi i}{3}}\right) \tag{4.2.16}
\end{equation*}
$$

The general solution of (4.2.11) is:

$$
\begin{align*}
f=c_{3} e^{k z}+c_{4} e^{-k z} & +\frac{e^{k z}}{2 k} \int e^{-k z}\left[c_{1} A i(u)+c_{2} A i\left(u e^{-\frac{2 \pi i}{3}}\right)\right] d z \\
& -\frac{e^{-k z}}{2 k} \int e^{k z}\left[c_{1} A i(u)+c_{2} A i\left(u e^{-\frac{2 \pi i}{3}}\right)\right] d z \tag{4.2.17}
\end{align*}
$$

Or in another form:

$$
\begin{align*}
f=c_{1} & {\left[\frac{e^{k z}}{2 k} \int e^{-k z} A i(u) d z-\frac{e^{-k z}}{2 k} \int e^{k z} A i(u) d z\right]+} \\
& +c_{2}\left[\frac{e^{k z}}{2 k} \int e^{-k z} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d z-\frac{e^{-k z}}{2 k} \int e^{k z} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d z\right]  \tag{4.2.18}\\
& +c_{3} e^{k z}+c_{4} e^{-k z}
\end{align*}
$$

In the general solution there are pure exponential terms. Hence, the solution has the form:

$$
\begin{equation*}
f_{a}=c_{1}\left[\frac{e^{k z}}{2 k} \int_{0}^{z} e^{-k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t-\frac{e^{-k z}}{2 k} \int_{0}^{z} e^{k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t\right]+c_{2} e^{-k z}+c_{3} e^{k \xi 4} . \tag{4.2.19}
\end{equation*}
$$

Since we want decay at infinity:

$$
\begin{align*}
\frac{c_{1}}{2 k} \int_{0}^{\infty} e^{-k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t+c_{3} & \triangleq \frac{c_{1}}{2 k} p_{1}\left(k, c, a, U_{0}\right)+c_{3}=0  \tag{4.2.20}\\
c_{3} & =-\frac{c_{1}}{2 k} p_{1} \tag{4.2.21}
\end{align*}
$$

For the case of constant current in the water $U_{w} \equiv U_{0}$, the solution for the water will be:

$$
\begin{equation*}
f_{w}=b_{1} e^{k z}+b_{2} e^{ \pm \sqrt{k\left(k+\frac{i}{\epsilon}\left(U_{0}-c\right)\right)} z} \triangleq b_{1} e^{k z}+b_{2} e^{B z} \tag{4.2.22}
\end{equation*}
$$

If $\Re(B)<0$, we need to choose $e^{-B}$ as the second independent Solution; and if $\Re(k)<$ 0 , we need to choose $e^{-k z}$ instead of $e^{k z}$.

And the derivatives are:

$$
\begin{gather*}
f_{w}^{\prime}=k b_{1} e^{k z}+B b_{2} e^{B z}  \tag{4.2.23}\\
f_{w}^{\prime \prime}=k^{2} b_{1} e^{k z}+B^{2} b_{2} e^{B z}  \tag{4.2.24}\\
f_{w}^{\prime \prime \prime}=k^{3} b_{1} e^{k z}+B^{3} b_{2} e^{B z} \tag{4.2.25}
\end{gather*}
$$

While at the interface it reduce to:

$$
\begin{array}{rr}
f_{w}(0)=b_{1}+b_{2}, & f_{w}^{\prime}(0)=k b_{1}+B b_{2}  \tag{4.2.26}\\
f_{w}^{\prime \prime}(0)=k^{2} b_{1}+B^{2} b_{2}, & f_{w}^{\prime \prime \prime}(0)=k^{3} b_{1}+B^{3} b_{2}
\end{array}
$$

In order to find the unknown coefficients, we need to apply the boundary conditions at the interface. In these boundary conditions the derivatives of $f$ play a main role. Thus,
we need to calculate $f_{w, a}(0), f_{w, a}^{\prime}(0), f_{w, a}^{\prime \prime}(0), f_{w, a}^{\prime \prime \prime}(0)$. This must be done carefully, using Leibbniz's rule for derivative under the integral operator.

$$
\begin{equation*}
\frac{d}{d z} \int_{f_{1}(z)}^{f_{2}(z)} g(t) d t=g\left(f_{2}(z)\right) f_{2}^{\prime}(z)-g\left(f_{1}(z)\right) f_{1}^{\prime}(z) \tag{4.2.27}
\end{equation*}
$$

Hence:

$$
\begin{align*}
& f_{a}^{\prime}=c_{1}\left[\frac{k e^{k z}}{2 k} \int_{0}^{z} e^{-k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t+\frac{e^{k z}}{2 k} e^{-k z} A i\left(u e^{-\frac{2 \pi i}{3}}\right)\right. \\
& \left.+\frac{k e^{-k z}}{2 k} \int_{0}^{z} e^{k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t-\frac{e^{-k z}}{2 k} e^{k z} A i\left(u e^{-\frac{2 \pi i}{3}}\right)\right]-k c_{2} e^{-k z}+k c_{3} e^{k z}= \\
& \quad=c_{1}\left[\frac{e^{k z}}{2} \int_{0}^{z} e^{-k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t+\frac{e^{-k z}}{2} \int_{0}^{z} e^{k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t\right]-k c_{2} e^{-k z}+k c_{3} e^{k z} \tag{4.2.28}
\end{align*}
$$

$$
\begin{array}{r}
f_{a}^{\prime \prime}=c_{1}\left[\frac{k e^{k z}}{2} \int_{0}^{z} e^{-k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t-\frac{k e^{-k z}}{2} \int_{0}^{z} e^{k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t\right. \\
\left.+A i\left(u e^{-\frac{2 \pi i}{3}}\right)\right]+k^{2} c_{2} e^{-k z}+k^{2} c_{3} e^{k z} \\
f_{a}^{\prime \prime \prime}=c_{1}\left[\frac{k^{2} e^{k z}}{2} \int_{0}^{z} e^{-k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t+\frac{k^{2} e^{-k z}}{2} \int_{0}^{z} e^{k t} A i\left(u e^{-\frac{2 \pi i}{3}}\right) d t+\right. \\
\left.+A i^{\prime}\left(u e^{-\frac{2 \pi i}{3}}\right) e^{\frac{\pi i}{6}}\left(\frac{k a}{\epsilon}\right)^{\frac{1}{3}}\right]-k^{3} c_{2} e^{-k z}+k^{3} c_{3} e^{k z} \tag{4.2.30}
\end{array}
$$

It will be helpful to mark a few values:

$$
\begin{align*}
u(0) & =u_{0}=\left(\frac{i \epsilon}{k a}\right)^{\frac{2}{3}} k\left(k+\frac{i}{\epsilon}\left(U_{0}-c\right)\right)  \tag{4.2.31}\\
A i\left(u(0) e^{-\frac{2 \pi i}{3}}\right) & =\tilde{A} i_{0}=A i\left(u_{0} e^{-\frac{2 \pi i}{3}}\right)  \tag{4.2.32}\\
A i(u(0)) & =A i_{0}=A i\left(u_{0}\right) \tag{4.2.33}
\end{align*}
$$

Now we can present the derivatives at the interface:

$$
\begin{gather*}
f_{a}(0)=c_{2}+c_{3}, f_{a}^{\prime}(0)=k\left(c_{3}-c_{2}\right), f_{a}^{\prime \prime}(0)=c_{1} \tilde{A} i_{0}+k^{2}\left(c_{2}+c_{3}\right) \\
f_{a}^{\prime \prime \prime}(0)=c_{1} \tilde{A} i_{0}^{\prime} e^{\frac{\pi i}{6}}\left(\frac{k a}{\epsilon}\right)^{\frac{1}{3}}+k^{3}\left(c_{3}-c_{2}\right)  \tag{4.2.34}\\
f_{w}(0)=b_{1}+b_{2}, \quad f_{w}^{\prime}(0)=k b_{1}+B b_{2}  \tag{4.2.35}\\
f_{w}^{\prime \prime}(0)=k^{2} b_{1}+B^{2} b_{2}, \quad f_{w}^{\prime \prime \prime}(0)=k^{3} b_{1}+B^{3} b_{2}
\end{gather*}
$$

Writing the boundary condition at the interface:

$$
\begin{equation*}
f_{a}(0)=f_{w}(0)=c-U_{0}=c_{2}+c_{3}=b_{1}+b_{2} \tag{4.2.36}
\end{equation*}
$$

$$
\begin{align*}
& f_{w}^{\prime}(0)+U_{w}^{\prime}(0)=f_{a}^{\prime}(0)+U_{a}^{\prime}(0) \Rightarrow k\left(c_{3}-c_{2}\right)=k b_{1}+B b_{2}-a_{a}  \tag{4.2.37}\\
& \mu\left(f_{a}^{\prime \prime}(0)+k^{2} f_{a}(0)+U_{a}^{\prime \prime}(0)\right)=\left(f_{w}^{\prime \prime}(0)+k^{2} f_{w}(0)+U_{w}^{\prime \prime}(0)\right) \Rightarrow \\
& \mu\left(c_{1} \tilde{A} i_{0}+k^{2}\left(c_{2}+c_{3}\right)+k^{2}\left(c_{2}+c_{3}\right)\right)=k^{2} b_{1}+B^{2} b_{2}+k^{2}\left(b_{1}+b_{2}\right) \Rightarrow  \tag{4.2.38}\\
& \mu c_{1} \tilde{A} i_{0}-k^{2} b_{1}-B^{2} b_{2}=k^{2}\left(c-U_{0}\right)(1-2 \mu)
\end{align*}
$$

We get a system of five linear equations with five unknowns $c_{1}, c_{2}, c_{3}, b_{1}, b_{2}$. After solving the system we have the value of the functions $f_{a}, f_{w}$ and its derivatives at the interface. Note that all of these constants are functions of the specific case which is defined by ( $R_{a}, R_{w}, a_{a}, U_{0}$ ) and the value of $\omega, k$. After we get these values, we can substitute them into the dynamic boundary condition and solve it, in order to find $\omega, k$.

$$
\begin{array}{r}
k f_{w}^{\prime}\left(c-U_{0}\right)+k f_{w} U_{w}^{\prime}+i R_{w}^{-1}\left(3 k^{2} f_{w}^{\prime}-f_{w}^{\prime \prime \prime}\right)-F=  \tag{4.2.39}\\
=\rho\left[k f_{a}^{\prime}\left(c-U_{0}\right)+k f_{a} U_{a}^{\prime}+i R_{a}^{-1}\left(3 k^{2} f_{a}^{\prime}-f_{a}^{\prime \prime \prime}\right)-F\right]+W k^{3} \text { at } z=0
\end{array}
$$

Practically, we are unable to get a simple dispersion relation for this case, thus in order to calculate it we need to do it numerically. The value of the constant $p_{1}$ (see equation (4.2.20)) can be calculated numerically and then the dispersion equation will be solved by a numeric solver for a nonlinear equation. The important thing is that the method of solution is very different from the numerical one. Thus, we can conclude that if the results will be similar it will validate the numerical model.

### 4.2.2 Convergence of the Chebyshev collocation method

As mentioned above, the computational domain is divided into three sub intervals: the water $\left[-z_{\infty}, 0\right]$, the air1 $\left[0, z_{1}\right]$ and the air2 interval $\left[z_{1}, z_{\infty}\right]$. In every one of these intervals we can control the number of collocation points (the grid). If the method converges, the change in the results should be minor when changing the number of collocation points. From numerical experiments, we learn that the interval air1 is the most important interval and therefore it requires a high resolution grid. We will show convergence when the number of collocation points in interval air1 $N_{a 1}$ equals those in the water $N_{w}$, and the number of collocation points in air2 will be $N_{a 2}=m N_{a 1}$ where $m$ is a parameter. See Table (4.2). As we can understand from the date in the table the process converge up to four significant digits for all the cases. Of course that the case of longer wavelength and high wind intensities required more grid points. Although
we can obtain differences between $m=1$ to $m=2.2$, this differences are in the order of $1 \%$.

### 4.2.3 Sensitivity to $z_{\infty}$

Since we practically need to use a finite interval for the numerical calculations, we have to show that our choice does not have a major effect on the results. The $z$ coordinate is normalized by $k_{0}$ in the form $z=z_{\text {dimensional }} k_{0}$. From the physical aspect, the value of $z_{\infty}$ should be related to the wave length. From the linear theory of water waves, it is known that after half of a wavelength the influence of the waves on the flow field is minor. The value which we use for most of our calculations is $z_{\infty}=10$, which means that $z_{\infty, \text { dimensional }}=\frac{10}{2 \pi} \lambda=1.591 \lambda$. In order to justify this value, we made a few runs with different values for $z_{\infty}$. Note that in this method, when we change the interval size we need to change the number of collocation points in order to keep on the fine grid in the critical region. See Table (4.3). From this table we can say that if there is an error it is in the fourth digit.

|  | $u_{*}=0.3 m / \sec \lambda=0.01 m$ | $u_{*}=0.8 m / \sec \lambda=0.01 m$ |
| :--- | :--- | :--- |
| $z_{\infty}=5$ | $1.36983130463987+0.01278003425023 \mathrm{i}$ | $1.68757278248872+0.27390650413015 \mathrm{i}$ |
| $z_{\infty}=10$ | $1.36986422621405+0.01276983682512 \mathrm{i}$ | $1.68758325820986+0.27389767316548 \mathrm{i}$ |
| $z_{\infty}=15$ | $1.36986417697524+0.01276981897276 \mathrm{i}$ | $1.68758030230838+0.27390093383573 \mathrm{i}$ |
| $z_{\infty}=20$ | $1.36986415990567+0.01276984833673 \mathrm{i}$ | $1.68757615130409+0.27390551097003 \mathrm{i}$ |

Table 4.3: Sensitivity to $z_{\infty}$, temporal case, "numerical" wind profile, exponential current. (values of $\omega$ )

רגישות ל־סz, התפתחות בזמן, פרופיל רוח ״נומרי״ וזרם אקספוננציאלי (ערכי $\omega$

### 4.2.4 Comparison with test case

In order to validate the numerical results, we want to compare them with an analytic solution. The case which we will compare them with is the case which was mentioned above. This case has a nontrivial solution, and thus the numerical solution is also not

| $u_{*}=0.3 \mathrm{~m} / \mathrm{sec}, \quad \lambda=0.01 m$ temporal case numeric wind profile exponential current |  |  |
| :--- | :--- | :--- |
| $N_{a 1}$ | $m=2.2$ | $m=1$ |
| 30 | $1.37010402587656+0.01263499596077 \mathrm{i}$ | $1.37019990579133+0.01261269160402 \mathrm{i}$ |
| 50 | $1.36986782416071+0.01277582929256 \mathrm{i}$ | $1.36986779786926+0.01277589534320 \mathrm{i}$ |
| 70 | $1.36986601469235+0.01277611104267 \mathrm{i}$ | $1.36986601478272+0.01277611106661 \mathrm{i}$ |
| 90 | $1.36986601371911+0.01277611191703 \mathrm{i}$ | $1.36986601371134+0.01277611202308 \mathrm{i}$ |
| 110 | $1.36986601240586+0.01277611222968 \mathrm{i}$ | $1.36986601277103+0.01277611185787 \mathrm{i}$ |
| 130 |  | $1.36986601187163+0.01277611317040 \mathrm{i}$ |


| $u_{*}=0.5 m /$ sec, $\quad \lambda=0.1 m$ temporal case numeric wind profile exponential current |  |  |
| :--- | :--- | :--- |
| $N_{a 1}$ | $m=2.2$ | $m=1$ |
| 120 | $0.87483141062881+0.17480983341175 \mathrm{i}$ | $0.87493320709386+0.17478722072537 \mathrm{i}$ |
| 140 | $0.87483171720101+0.17480776493565 \mathrm{i}$ | $0.87486612961292+0.17480001977824 \mathrm{i}$ |
| 160 | $0.87483188719137+0.17480774370673 \mathrm{i}$ | $0.87484264656896+0.17480538277516 \mathrm{i}$ |
| 180 | $0.87483279917512+0.17480688132684 \mathrm{i}$ | $0.87483464528386+0.17480712587869 \mathrm{i}$ |
| 200 | $0.87483175839842+0.17480724161745 \mathrm{i}$ | $0.87483427921663+0.17480516926405 \mathrm{i}$ |


| $u_{*}=1 m / \mathrm{sec}, \quad \lambda=0.2 m$ temporal case numeric wind profile exponential current |  |  |
| :--- | :--- | :--- |
| $N_{a 1}$ | $m=2.2$ | $m=1$ |
| 120 | $0.47142417036677+1.12889159237943 \mathrm{i}$ | $0.47349531861018+1.22330241493232 \mathrm{i}$ |
| 140 | $0.46563247157728+1.13326078539474 \mathrm{i}$ | $0.47852328385567+1.12123661924386 \mathrm{i}$ |
| 160 | $0.46506653333590+1.13087658893075 \mathrm{i}$ | $0.47825231439927+1.10088627966011 \mathrm{i}$ |
| 180 | $0.46531992042561+1.12935842506008 \mathrm{i}$ | $0.47431190399082+1.10397392317904 \mathrm{i}$ |
| 200 | $0.46526678569868+1.12910983900462 \mathrm{i}$ | $0.46777840741684+1.12542290998380 \mathrm{i}$ |
| 220 | $0.46554496634933+1.12828921832640 \mathrm{i}$ | $0.46349077942169+1.14126307853594 \mathrm{i}$ |
| 240 | $0.46517456226768+1.12930446359643 \mathrm{i}$ | $0.46306242326445+1.14140291289982 \mathrm{i}$ |


| $u_{*}=0.5 m /$ sec,$\quad \lambda=0.1 m$ spatial case numeric wind profile exponential current |  |  |
| :--- | :--- | :--- |
| $N_{a 1}$ | $m=2.2$ | $m=1$ |
| 120 | $1.73821643904331-0.54907567574767 \mathrm{i}$ | $1.73816601034713-0.54929404443106 \mathrm{i}$ |
| 140 | $1.73820819328841-0.54905128308376 \mathrm{i}$ | $1.73818431297562-0.54913186335472 \mathrm{i}$ |
| 160 | $1.73820756864928-0.54905212376898 \mathrm{i}$ | $1.73819902935100-0.54908010735431 \mathrm{i}$ |
| 180 | $1.73820751242298-0.54905337641793 \mathrm{i}$ | $1.73820423621426-0.54906078085848 \mathrm{i}$ |
| 200 | $1.73820621846891-0.54905233666072 \mathrm{i}$ | $1.73820455823729-0.54905742974889 \mathrm{i}$ |


| $u_{*}=1 \mathrm{~m} / \mathrm{sec}, \quad \lambda=0.2 m \quad$ spatial case numeric wind profile exponential current |  |  |
| :--- | :--- | :--- |
| $N_{a 1}$ | $m=2.2$ | $m=1$ |
| 120 | $1.45509895054817+0.17847430498794 \mathrm{i}$ | $1.44599590306100+0.18412258134930 \mathrm{i}$ |
| 140 | $1.46114514767122+0.17957479792305 \mathrm{i}$ | $1.46201412747482+0.17714187966656 \mathrm{i}$ |
| 160 | $1.46133041441920+0.17871796801299 \mathrm{i}$ | $1.46531718279471+0.17653390423403 \mathrm{i}$ |
| 180 | $1.46144308755361+0.17867789454668 \mathrm{i}$ | $1.46458249190827+0.17704152485829 \mathrm{i}$ |
| 200 | $1.46147219135630+0.17867063841887 \mathrm{i}$ | $1.46205719912314+0.17841340733186 \mathrm{i}$ |
| 220 | $1.46154738497208+0.17863415163699 \mathrm{i}$ | $1.46017536775865+0.17935843070790 \mathrm{i} 1$ |
| 240 | $1.46145750594983+0.17868915488061 \mathrm{i}$ | $1.46010173867998+0.17936518635012 \mathrm{i}$ |

Table 4.2: Convergence of the Chebyshev collocation method for various wind intensity and wavelength, temporal/spatial case. Where $N_{a 1}=N_{w}, N_{a 2}=m N_{a 1}$. (values of $\omega, k$ )
התכנסות שיטת הקולוקציות של צ׳בישב לעוצמות רוח ואורכי גל שונים עבור התפתחות בזמן/במרחב, כאשר $N_{a 1}=N_{w}, \quad N_{a 2}=m N_{a 1}(\omega, k$ (ערכי
so trivial. As mentioned above, if we want to calculate the eigenvalues by the analytic solution we need to use numerical methods in order to calculate the integral of $p_{1}$ and solve the dynamic boundary condition equation. Thus, the solution is not purely analytical. Such a comparison has strong meaning, because the methods which were used to obtain the results are completely different. The comparison was done for the spatial case, as well as for the temporal case. The current was constant with the value of $U_{w}=0.5 u_{*}$ and the velocity slope of the wind was $a_{a}=\frac{u_{*}}{\nu_{a}}$. These conditions are similar to those which were used in the lin-log profile. The results are described in the Table (4.4). The maximum error in this comparison is $2 \%$ while in most of the cases the error is smaller than $1 \%$.

| Temporal case |  |  |  |  |  |  | $\lambda=0.001 m$ | $\lambda=0.1 m$ | $\lambda=0.2 m$ |
| :--- | :--- | :--- | :--- | :--- | :---: | :---: | :---: | :---: | :---: |
| $u_{*}=0.1 \mathrm{~m} / \mathrm{sec}$ | Numerical | $1.0705-1.7431 \mathrm{e}-002 \mathrm{i}$ | $1.1246+7.1116 \mathrm{e}-003$ | $1.0908+9.1061 \mathrm{e}-003 \mathrm{i}$ |  |  |  |  |  |
|  | Analytical | $1.0705-1.7431 \mathrm{e}-002 \mathrm{i}$ | $1.1246+7.1115 \mathrm{e}-003 \mathrm{i}$ | $1.0908+9.1070 \mathrm{e}-003 \mathrm{i}$ |  |  |  |  |  |
| $u_{*}=0.5 \mathrm{~m} / \mathrm{sec}$ | Numerical | $1.3628-1.6102 \mathrm{e}-002 \mathrm{i}$ | $1.7260+2.2254 \mathrm{i}$ | $1.7684+2.5485 \mathrm{ei}$ |  |  |  |  |  |
|  | Analytical | $1.3628-1.6102 \mathrm{e}-002 \mathrm{i}$ | $1.7262+2.2254 \mathrm{i}$ | $1.7680+2.5489 \mathrm{i}$ |  |  |  |  |  |
| $u_{*}=1 \mathrm{~m} / \mathrm{sec}$ | Numerical | $1.7132-2.8672 \mathrm{e}-005 \mathrm{i}$ | $4.7194 \mathrm{e}-001-7.6548 \mathrm{i}$ | $5.3226+8.4767 \mathrm{i}$ |  |  |  |  |  |
|  | Analytical | $1.7132-2.9147 \mathrm{e}-005 \mathrm{i}$ | $4.6973 \mathrm{e}-001-7.6153 \mathrm{i}$ | $5.3219+8.4752 \mathrm{i}$ |  |  |  |  |  |


| Spatial case |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  |  | $T=0.00145 \mathrm{sec}$ | $T=0.249 \mathrm{sec}$ | $T=0.356 \mathrm{sec}$ |
| $u_{*}=0.1 \mathrm{~m} / \mathrm{sec}$ | Numerical | $9.5422 \mathrm{e}-01+1.0392 \mathrm{e}-02 \mathrm{i}$ | 2.8359e-01-2.7626i |  |
|  | Analytical | $9.5419 \mathrm{e}-001+1.0450 \mathrm{e}-002 \mathrm{i}$ | $2.8359 \mathrm{e}-001-2.7626 \mathrm{i}$ | $1.2187 \mathrm{e}-002+7.8251 \mathrm{e}-003 \mathrm{i}$ |
| $u_{*}=0.5 \mathrm{~m} / \mathrm{sec}$ | Numerical | $7.9595 \mathrm{e}-01+5.5140 \mathrm{e}-03 \mathrm{i}$ | 1.7019e+01-2.7862i | $3.5163 \mathrm{e}+01-5.3232 i$ |
|  | Analytical | $7.9592 \mathrm{e}-001+5.5216 \mathrm{e}-003 \mathrm{i}$ | $1.7018 \mathrm{e}+001-2.7872 \mathrm{i}$ | $3.5162 \mathrm{e}+001-5.3248 \mathrm{i}$ |
| $u_{*}=1 \mathrm{~m} / \mathrm{sec}$ | Numerical | $6.6253 \mathrm{e}-01$ - 8.6724e-03i | 6.3738e+01-6.9994i | 1.2819e+02-1.3851e+01i |
|  | Analytical | $6.6255 \mathrm{e}-001$-8.6253e-003i | $6.3738 \mathrm{e}+001-6.9966 \mathrm{i}$ | $1.2815 \mathrm{e}+002-1.3909 \mathrm{e}+001 \mathrm{i}$ |

Table 4.4: Comparison with test case, temporal and spatial case for various wind intensity and wavelength/waveperiod. (values of $\omega, k$ ) השוואה עם מקרה הבוחן עבור עוצמות רוח ואורכי גל/זמני מחזור שונים (ערכי $\omega$ )

### 4.2.5 Comparison with previous studies

The last stage in the validation process will be a comparison with previous calculations done by Kawai [11], Van-Gastel [26] and Tsai [22]. They all compute similar values for the temporal growth rate, but for a very small domain in the $u_{*}, \omega$ space. A figure which is similar to Fig. 2 in [22] is presented in Fig. (4.2).


Figure 4.2: Energy growth rate vs. wavenumber temporal case. Comparison with previous studies. (similar to Fig. 2 in Tsai and Lin [22]).

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קצב גידול האנרגיה כתלות במספר הגל, התפתחות בזמן, השוואה עם עבודות קודמות 
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In Fig. (4.2) there are symbols of our calculations and symbols of Tsai and Lin's calculations taken From Fig. 2 in [22]. We can see that the results of Tsai and Lin [22] are similar to our results, the differences are up to $5 \%$ but usually it is in the order of $1 \%$. For the spatial case, we are not familiar with previous studies which we could compare it with. Finally, we can say that for the range of $\lambda \in(0.001,0.2) m, u_{*} \in(0,1) \mathrm{m} / \mathrm{sec}$ it seems that the numerical results are valid. In most of the cases the accuracy is in the first four digits while in the difficult cases it is only in the first two significant digits.

## Chapter 5

## Results and Discussion

In this chapter we present the results of our calculations. The results are the calculated eigenvalues for each scenario. A scenario is defined by: wind and current profiles, friction velocity and wavelength/waveperiod for the temporal/spatial case respectively. We calculated the eigenvalues in the range $0.001 m<\lambda_{0}<0.2 m, 0.1 m / s e c<u_{*}<$ $1 \mathrm{~m} / \sec$ for the following cases: the inviscid model using the "one" and the "numerical" profiles for both temporal and spatial cases, the viscous model with the lin-log and "numerical" profiles for both temporal and spatial cases. From the imaginary part of the eigenvalue we can obtain the growth rate, and from the real part we can obtain the phase velocity. The results are presented graphically.

### 5.1 Results for the inviscid model, Rayleigh's Equation

Generally there is a very large difference between the results when using the "one" profile and those when using the "numerical" profile. These differences come from the different behavior of the mean flow derivatives. In Rayleigh's equation the resulting growth rate is proportional to the mean flow curvature $-U^{\prime \prime}\left(z_{c}\right)$. The curvature of these two profiles ("one","numerical") is completely different. The critical point $z_{c}$ when $U(z)=c$, plays a main role in this model. The reason for the importance of this point is that the equation is close to singular near this point; since the coefficient of the dominant derivative is close to zero. Another interesting quality of the solution is the symmetry relative to the real axis, which means that if $\omega$ is an eigenvalue then $\bar{\omega}$ is also an eigenvalue. When using the "one" profile the drift velocity is zero, hence
there is always a critical point. When using the "numerical" profile the drift velocity is $U_{0}=0.5 u *$, hence theoretically it is possible that such a critical point does not exist in the air because $U_{0}>c_{0}$. Practically, we never obtained such a case, because the calculated phase velocity $c$ is always larger than the drift current. In the case of the "one" profile there is no current, thus the results are governed by the wind. In this case it is easier to understand the results. One can see that the curves of the growth rate see Fig. (5.1) have one maximum point. The value of the wavelength $\lambda$ at this maximum point increases when the wind intensity increases, but the maximum growth rate does not increase monotonically when increasing the wind intensity, see also Figs. (A.5),(A.6). This behavior is similar in the temporal case and in the spatial case see Fig.(5.3). The resulting phase velocity does not deviate much from the phase velocity of the reference problem, see Figs. (A.1),(A.3) or Fig.(5.2),(5.4). In Fig. (A.4),(A.2) it is shown that the curves of constant $u_{*}$ at the space of the normalized complex eigenvalue generate a spiral shape for both the spatial and the temporal case.

When using the "numerical" profile the results are very different. The curves of the growth rate have a much more complex shape see Fig.(5.6),(5.8). The reason for this complex shape is mainly because the curvature of that profile has a maximum near the interface. The presence of the exponential current makes this scenario very complicated regarding the phase velocity. This is because the behavior of waves on such a shear current is not that trivial. We can see that the short waves are strongly influenced by the current and this causes large deviation from the reference phase velocity, see Fig.(5.7),(5.9) or (A.11),(A.14). These deviations are very similar in the spatial and in the temporal case, see Fig.(A.18). The eigenvalue picture in that case (see Fig.(A.16),(A.13)) is again very complicated, but we can obtain patterns which are similar to the patterns in the results with the "one" profile.

When solving the problem of finding the eigenvalue, we also find the eigenfunction which corresponding to this eigenvalue. The eigenfunction $f(z)$ is a complex function which defines the whole flow field (velocity, pressure). Hence it is interesting to observe its behavior. Since the governing equation is close to singularity near the critical point, we expect to get a boundary layer at that zone. When looking at the absolute value of the eigenfunction, see Fig.(5.11), we can see that sometimes it has two extremum points and sometimes only one. These extremum points do not have a special physical
meaning because they are very close to the interface, and since the interface is the mean water surface, the fluid near this interface can be either water or air. In the same figure we can see that there are very large derivatives near the interface, and then ask how the viscosity will influence such large derivatives.

Another interesting point is how the surface tension influences the growth rate. As we can see in Fig.(5.5) which shows the growth rate for a wide range of wavelengths and wind intensities, the surface tension starts to play a significant role only for waves with wavelength shorter then 3 cm .


Figure 5.1: Temporal growth rate vs. wavelength for various values of $u_{*}$, inviscid model, "one" profile
קצב גידול בזמן כתלות באורך הגל עבור המודל הבלתי צמיג ופרופיל "אחד"


Figure 5.2: Phase velocity vs. wavelength for various values of $u_{*}$, inviscid model, "one" profile, temporal case

מהירות הפזה כתלות באורך הגל עבור המודל הבלתי צמיג ופרופיל ״אחד״, התפתחות בזמן


Figure 5.3: Spatial growth rate vs. waveperiod for various values of $u_{*}$, inviscid model, "one" profile

קצב גידול במרחב כתלות בזמן המחזור עבור המודל הבלתי צמיג ופרופיל "אחד"


Figure 5.4: Phase velocity vs. waveperiod for various values of $u_{*}$, inviscid model, "one" profile, spatial case
מהירות הפזה כתלות בזמן המחזור עבור המודל הבלתי צמיג ופרופיל ״אחד״, התפתחות במרחב


Figure 5.5: The influence of the surface tension on the growth rate for various values of $u_{*}$, inviscid model, "one" profile, temporal case

השפעת מתח הפנים על קצב הגידול עבור המודל הבלתי צמיג ופרופיל ״אחד״, התפתחות


Figure 5.6: Temporal growth rate vs. wavelength for various values of $u_{*}$, inviscid model, "numerical" profile קצב גידול בזמן כתלות באורך הגל עבור המודל הבלתי צמיג ופרופיל ״נומרֵ"


Figure 5.7: Phase velocity vs. wavelength for various values of $u_{*}$, inviscid model, "numerical" profile, temporal case

מהירות הפזה כתלות באורך הגל עבור המודל הבלתי צמיג ופרופיל ״נומרי״, התפתחות


Figure 5.8: Spatial growth rate vs. waveperiod for various values of $u_{*}$, inviscid model, "numerical" profile קצב גידול במרחב כתלות בזמן המחזור עבור המודל הבלתי צמיג ופרופיל "נומרי"


Figure 5.9: Phase velocity vs. waveperiod for various values of $u_{*}$, inviscid model,
"numerical" profile, spatial case
מהירות הפזה כתלות בזמן המחזור עבור המודל הבלתי צמיג ופרופיל ״נומרֵ״, התפתחות במרחב


Figure 5.10: Vertical structure of the eigenfunction and its derivative at $\lambda=5 \mathrm{~cm}$ for various values of $u_{*}$, inviscid model, "numerical" profile, temporal case פרופיל הפונקציה העצמית ונגזרתה עבור אורך גל $\lambda=5 c m, ה מ ו ד ל ~ ה ב ל ת י ~ צ מ י ג, ~ פ ר ו פ י ל ~$ "נומר", התפתחות בזמן


Figure 5.11: Vertical structure of the eigenfunction and its derivative (zoom in) at $\lambda=5 \mathrm{~cm}$ for various values of $u_{*}$, inviscid model, "numerical" profile, temporal case פרופיל הפונקציה העצמית ונגזרתה (הגדלה) עבור אורך גל $\lambda=5 \mathrm{~cm}$, ,המודל הבלתי צמיג, פרופיל ״נומר״״, התפתחות בזמן

### 5.2 Results for the viscous model, Orr-Sommerfeld Equation

For the viscous model the results were calculated with the "lin-log" and the "numerical" profiles. The results for these two profiles are qualitatively very similar. The mechanism of growth in this case is more complex than the mechanism for the inviscid case. The critical point does not play a main role anymore. The curves of constant wind intensity at the $\beta, \lambda$ space have one maximum point which is usually at the range of a few centimeters, see Figs.(5.12),(5.16) or Figs.(5.14),(5.18). Note that in the viscous case the dissipation due to viscosity is included, which is very dominant in short waves. During the calculations we discovered a second unstable mode. As already mentioned, we talk about an eigenvalue problem. For such problems there are many eigenvalues, but we are interested only in the unstable modes. For all of the cases which we solved until now and in all of the articles which were mentioned, there is only one growing wave for a specific scenario $u_{*}, \lambda$. The presence of the second mode (we call it branch2) starts at a specific wind intensity $u_{*}$; for the case of $U_{0}=0.5 u_{*}$ it appears at approximately $\sim u_{*}=0.55 \mathrm{~m} / \mathrm{sec}$. The values of the growth rates are very similar to those of branch1, but the maximum points refer to a larger wavelength with respect to branch1, see Figs.(A.23),(A.39) or similar figures. When looking at the phase velocity the values of the second branch are very different, they have a much lower phase velocities, see Figs.(5.13),(5.15) or Figs.(5.17),(5.19). It seems that the curves of the phase velocity approach a constant value when increasing the wavelength/waveperiod. This behavior is expected, but the resulting values of the phase velocities at large wavelength are significantly slower than the reference problem. These results are very unexpected because of the presence of the wind and the current. In the spatial case we only present branch1, which is the regular mode. The reason for this is numerical difficulties in the searching process of the algorithm. The origin of these difficulties is at the limitation of the method to deal with large wavelength, but as part of the searching process the temporary guess can jump to the illegal zone at the $k$ space. We believe that there is a second unstable mode in the spatial case as well, see Fig.(5.24). There are two interesting points in the growth curves for both the spatial and the temporal case. The first is the point of maximum growth, which is interesting because we expect that it will play
a dominant role. The second is the point of zero growth, which has the meaning of the shortest wave that can exist. We can see the behavior of these important points for each case, see for example Fig.(A.23),(A.24). For the temporal case we obtain that the ratio of $\frac{c}{u_{*}}$ at the point of maximum growth and the point of the neutral wave is almost constant at the second branch.

When looking in the vertical structure of the eigenfunction see Fig.(5.20) we obtain structure which is very similar to the one in the inviscid case. The major difference is at high wind intensities where the critical point is usually in the water see Fig.(5.21). As already mention there are two unstable modes at high wind intensities, although the significant differences in the phase velocity and in the growth rate the structure of the eigenfunction is very similar see Fig.(5.22),(5.23).


Figure 5.12: Temporal growth rate vs. wavelength for various values of $u_{*}$, viscous model, lin-log profile

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קצב גידול בזמן כתלות באורך הגל עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי
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Figure 5.13: Phase velocity vs. wavelength for various values of $u_{*}$, viscous model, lin-log profile, temporal case

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מהירות הפזה כתלות באורך הגל עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי , התפתחות בזמן
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Figure 5.14: Spatial growth rate vs. waveperiod for various values of $u_{*}$, viscous model, "lin-log" profile


Figure 5.15: Phase velocity vs. waveperiod for various values of $u_{*}$, viscous model, "lin-log" profile, spatial case מהירות הפזה כתלות בזמן המחזור עבור המודל הצמיג ופרופיל ליניארי לוגריתמי, התפתחות במרחב


Figure 5.16: Temporal growth rate vs. wavelength for various values of $u_{*}$, viscous model, numerical profile


Figure 5.17: Phase velocity vs. wavelength for various values of $u_{*}$, viscous model, "numerical" profile, temporal case מהירות הפזה כתלות באורך הגל עבור המודל הצמיג ופרופיל ״נומר״״, התפתחות בזמן


Figure 5.18: Spatial growth rate vs. waveperiod for various values of $u_{*}$, viscous model, "numerical" profile


Figure 5.19: Phase velocity vs. waveperiod for various values of $u_{*}$, viscous model, "numerical" profile, spatial case מהירות הפזה כתלות בזמן המחזור עבור המודל הצמיג ופרופיל ״נומרי״, התפתחות במרחב


Figure 5.20: Vertical structure of the eigenfunction and its derivative for $\lambda=5 \mathrm{~cm}$ and various values of $u_{*}$, viscous model, "numerical" profile, temporal case

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פרופיל הפונקציה העצמית ונגזרתה עבור אורד גל \lambda=5cm ,המודל הצמיג, פרופיל 
    "נומר",, התפתחות בזמן
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Figure 5.21: Vertical structure of the eigenfunction and its derivative (zoom in) for $\lambda=5 \mathrm{~cm}$ and various values of $u_{*}$, viscous model, "numerical" profile, temporal case פרופיל הפונקציה העצמית ונגזרתה (הגדלה) עבור אורך גל $\lambda=5 \mathrm{~cm}, ה מ ו ד ל ~ ה צ מ י ג, ~$ פרופיל ״נומר״״, התפתחות בזמן


Figure 5.22: Vertical structure of the eigenfunction and its derivative for $\lambda=2 c m, u_{*}=0.8 \mathrm{~m} / \mathrm{sec}$, viscous model, "numerical" profile, temporal case פרופיל הפונקציה העצמית ונגזרתה עבור אורך גל $\lambda=2 c m$ ומהירות חיכוך u* $=0.8 \mathrm{~m} / \mathrm{sec}$


Figure 5.23: Vertical structure of the eigenfunction and its derivative (zoom in) for
$\lambda=2 c m, u_{*}=0.8 \mathrm{~m} / \mathrm{sec}$, viscous model, "numerical" profile, temporal case פרופיל הפונקציה העצמית ונגזרתה (הגדלה) עבור אורך גל $\lambda=5 \mathrm{~cm}$ ומהירות חיכוך $u_{*}=0.8 \mathrm{~m} / \mathrm{sec}$


Figure 5.24: The square norm of the dynamic boundary condition at the $k$ space for $\lambda=2 \mathrm{~cm}, u_{*}=0.8 \mathrm{~m} / \mathrm{sec}$, viscous model, "numerical" profile, spatial case מפת הנורמה בריבוע של התנאי הדינאמי במישור מספר הגל עבור , $\lambda=2 \mathrm{~cm}, u_{*}=0.8 \mathrm{~m} / \mathrm{sec}$

### 5.3 Comparison between Rayleigh and Orr-Sommerfeld

Two questions form the basis of this section. The first deals with the significance of viscosity for such a problem, and the second question is whether Rayleigh's equations are asymptotically an approximation for $R e \rightarrow \infty$ of the Orr-Sommerfeld equations. The possibilities of comparing between these two models in the range of wavelength $0.001 m<\lambda<0.2 m$ were opened when we started to use the "numerical" profile, since the comparison using the other profile was very problematic. We compare the results of these two models using the following definition:

$$
\begin{array}{r}
G=k f_{w}^{\prime}\left(c-U_{0}\right)+k f_{w} U_{w}^{\prime}+i R_{w}^{-1}\left(3 k^{2} f_{w}^{\prime}-f_{w}^{\prime \prime \prime}\right)-k F- \\
-\rho\left[k f_{a}^{\prime}\left(c-U_{0}\right)+k f_{a} U_{a}^{\prime}+i R_{a}^{-1}\left(3 k^{2} f_{a}^{\prime}-f_{a}^{\prime \prime \prime}\right)-k F\right]+W k^{3} \text { at } z=0 \tag{5.3.1}
\end{array}
$$

Where $G(\omega, k)$ is the dynamic boundary condition. Since it is a complex function and $\omega, k$ are also complex numbers, we can plot the square norm of $G(\omega, k)$ at the space of one of these two variables for the temporal and spatial case, respectively. Where of course, $\|G\|^{2}=G \bar{G}$. This tool enables us to compare not only the eigenvalue or the eigenfunction but also the patterns of these surfaces. In order to simplify the analysis, we try to separate the effect of each component - the wind or the current. In Figs.(5.25),(5.26), we can find a set of six figures which compare the models for various cases, where the right column is the viscous case and the left one is the inviscid case. The first row is when there is only exponential current (no wind); the second row is when the current is constant and the wind is given by the "numerical" profile; the third row is with both "numerical" wind and exponential current. Both sets of figures are for the same wavelength $\lambda=5 \mathrm{~cm}$, but with different wind intensities $u_{*}=0.3,0.8 \mathrm{~m} / \mathrm{sec}$. In the first set, when $u_{*}=0.3 \mathrm{~m} / \sec$ we can see that the real part in all of the figures is approximately the same. On the other hand, at the imaginary part there are differences of about $\sim 100 \%$ where the viscous case estimates larger growth, except for the case of no wind. However, we can obtain a similar pattern. In the second set, when $u_{*}=0.8 \mathrm{~m} / \sec$ there are more differences between the models. In the case of no wind, we obtain that the viscous model can produce an unstable mode, this instability is due to the current only. On the other hand, the inviscid model can not produce growth but it is affected by the current; note that the second point is not a solution but only a local minimum. In the case of constant current, the inviscid model has two solutions, but
only one indicating growth. The second solution has a slower phase velocity, where the viscous model produces one growing mode with almost twice as much growth. In the last case, where the current and the wind are both affecting the problem, the differences become very large and it seems that there is no connection between the plots. After comparing the dynamic boundary condition plots we also compare the resulting eigenfunctions, see Figs.(5.27),(5.28). In these figures, we observe similar patterns in the eigenfunctions structure, but larger differences in the values of the functions. Finally, we can say that for this specific problem we find significant differences between these two models. These differences are due to two main reasons. The first is the mathematical reason: the viscous model leads to a fourth order equation while the inviscid one leads to a second order equation, which means that the viscous solution satisfies two boundary conditions which the inviscid solution can not satisfy, and one more boundary condition (dynamic) which can be significantly different than the one of the inviscid model. The second reason is that it is an interface problem and there are a few more forces which play a main roles. We can also say that these differences become larger when increasing the wind intensity. For these reasons, the inviscid model can produce results which seem problematic from the physical aspect. For example, the inviscid model can not produce growth for two cases in which the viscous model predicts growth or decay. The first is for the linear wind profile, where the solution for Rayleigh's equation is trivial - since $U^{\prime \prime} \equiv 0$. The second is for a wave traveling against the wind, where the inviscid model predicts no decay - because $U^{\prime \prime}>0$.


Figure 5.25: Comparison between the Rayleigh and Orr-Sommerfeld, plots of the dynamic boundary condition for $\lambda=5 c m, u_{*}=0.3 \mathrm{~m} / \mathrm{sec}$, temporal case.(a)-Inviscid model, no wind, exponential current,(b)-Viscous model, no wind, exponential current,
(c)-Inviscid model, "numerical" wind profile, constant current,(d)-Viscous model, "numerical" wind, constant current, (e)-Inviscid model, "numerical" wind profile, exponential current,(f)-Viscous model, "numerical" wind, exponential current השוואה בין המודל הצמיג והמודל הבלתי צמיג עבור $u_{*}=0.3 m / \sec \lambda=5 c m, ~ ה ת פ ת ח ו ת ~$


Figure 5.26: Comparison between the Rayleigh and Orr-Sommerfeld plots of the dynamic boundary condition for $\lambda=5 \mathrm{~cm}, u_{*}=0.8 \mathrm{~m} / \mathrm{sec}$, temporal case.(a)-Inviscid model, no wind, exponential current,(b)-Viscous model, no wind, exponential current,
(c)-Inviscid model, "numerical" wind profile, constant current,(d)-Viscous model, "numerical" wind, constant current, (e)-Inviscid model, "numerical" wind profile, exponential current,(f)-Viscous model, "numerical" wind, exponential current השוואה בין המודל הצמיג והמודל הבלתי צמיג עבור



Figure 5.27: Comparison of the resulted eigenfunctions from the inviscid/viscous model for $\lambda=2 \mathrm{~cm}$, temporal case. (a) $-u_{*}=0.3 \mathrm{~m} / \mathrm{sec}$, (b) $-u_{*}=0.8 \mathrm{~m} / \mathrm{sec}$ השוואה של הפונקציות העצמיות המתקבלות במודלים השונים, $\lambda=2 c m, ~ ה ת פ ת ח ו ת ~$ בזמן


Figure 5.28: Comparison of the resulted eigenfunctions from the inviscid/viscous
model for $\lambda=15 \mathrm{~cm}$, temporal case. (a) $-u_{*}=0.3 \mathrm{~m} / \mathrm{sec}$, (b) $-u_{*}=0.8 \mathrm{~m} / \mathrm{sec}$ השוואה של הפונקציות העצמיות המתקבלות במודלים השונים, $\lambda=15 \mathrm{~cm}, ~ ה ת פ ת ח ו ת ~$

### 5.4 Comparison with experiments

When trying to compare with past experiments, we find that there are not so many experimental studies which publish data in a form that we can compare with. One of the comprehensive experimental studies is Larson and Wright [13]. They study temporal growth of gravity-capillary waves. Larson and Wright publish all of their results, including the air profile measurement in a form that enables comparison. In Figs.(5.29),(5.30) we plot our results in figures which are similar to those of Larson and Wright at two wavelengths. In these figures there are six lines: the two blue lines represent the results of the viscous model using "numerical" profile at the air and exponential current, the red line represents the results of the viscous model using "numerical" profile at the air and no current, the magenta line represents the results of the inviscid model using "numerical" profile at the air and exponential current, the green line represents the results of Valenzuela [24] and the black line represents the experimental results of Larson and Wright. We can state that the calculated results are far from being close to the experimental measurements. There are a lot of possible reasons why the results turn out to be so different. For example, we can say that the wind and current profile may be very different in reality and maybe we should have coupled the problem of the base flow with the waves problem. Another reason can be the effect of turbulence on the growth mechanism.


Figure 5.29: Comparison of calculations with experimental result of [13] for $\lambda=4.05 \mathrm{~cm}$, temporal case

השוואה של תוצאות חישובים שונים עם ניסויים $\lambda=4.05 \mathrm{~cm}, ~ ה ת פ ת ח ו ת ~ ב ז מ ן ~$


Figure 5.30: Comparison of calculations with experimental result of [13] for

$$
\lambda=6.98 \mathrm{~cm} \text {, temporal case }
$$

### 5.5 Summary and conclusions

- A full formulation of the linear stability problem of water waves in the presence of a shear flow is presented for both an inviscid and viscid fluids.
- Robust solvers for both models were develop. For the viscous model we expand the domain of calculations (in $\lambda, u_{*}$ space) by factor of six, compared with previous studies.
- The calculations were done with a range of wind profiles. The sensitivity of the results to the chosen profile were confirmed. We also propose a new profile that is based on the Prandtl's mixing length theory.
- The current affect strongly on the calculated phase velocity but has a minor effect on the growth rate.
- The effect of surface tension on the growth rate (in the inviscid case) were found to be important for waveslength that is shorter than 3 cm .
- The presence of a second unstable mode was obtained for the viscous model at high wind intensities, for profiles with shear current.
- A comparison between the inviscid and the viscous model was done. Significant differences between these two models were obtained.


## Chapter 6

## Future research

### 6.1 Experimental aspects

The disagreement between the theoretical results of this study and the experimental results of Larson and Wright calls for a comprehensive experimental study. A cooperation with the experimental group of Prof. Lev Shemer in Tel-Aviv university has already begun. A close cooperation can lead to a better agreement between calculated and measured results. The first issue to be dealt with is the mean wind and current profiles. If we will know the values of the drift current, the current and the wind close to the interface, we can build a profile and use it to make calculations. Another subject is the development of boundary layer along the tank, here we can build a quasi steady model that, for example, calculates the growth at every cross section and integrates it along the tank.

### 6.2 Linear theoretical aspects

- Imposing all of the interface boundary conditions at the real interface $z=\eta$. Imposing a more accurate boundary condition can be important since the linear interface $z=0$ is either water or air and the interface plays a main role in the problem. It can also provide a better physical meaning to the values of the auxiliary function. This can be done using coordinate transformation.
- The mean flow profile is a main issue in this model. The results of this study call
for a better understanding of the mean flow profile above the surface waves. Currently, the assumption is that the flow above waves is similar to the flow above a flat plate; this assumption does not fit to the results of Larson and Wright's measurements. Another aspect is the dominance of the very thin layer near the surface, a region in which it is almost impossible to make measurements. The task will be to build a profile for the air as well as for the water which will fit the observations and will be based on a physical theory.
- The calculations were done using standard computational tools. There is a possibility to expand the domain of numerical calculations. Such expanding is important if we are interested in seeing the results of the viscous case for large wavelengths. A more robust numerical model can be built. We can improve the precision for each number, as well as the specific method of applying the numerical method. Such an improved model can be more accurate and with a shorter runtime.
- Turbulence is a main issue and there are many studies that calculate the growth rate using a full turbulent approach. The task can be to introduce the turbulent fluctuations into the linear model of Miles', using one of the simple eddy viscosity formulation.


### 6.3 Nonlinear theoretical aspects

- Study the effect of wind when there are two or more wavelengths. This aspect can be important when trying to predict the wave growth in the experiment.
- Study the effect of wind on left-right asymmetry of the wave shape. It is known from observations that there is such asymmetry in wind waves. Such asymmetry can cause a better description of the wave shape and thus more accurate values when calculating the growth rates.
- Couple the problem of mean flow profile to the problem of wave generation. Such coupling can be the answer to the sensitivity of the models to the mean flow profile.
- Taking the full energy and momentum balance between the atmosphere and ocean
into account and trying to predict how much momentum is transferred to the current and to the waves.


## Appendix A

## Additional Results



Figure A.1: Normalized phase velocity vs. wavelength for various values of $u_{*}$, inviscid model, "one" profile, temporal case

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Figure A.2: Eigenvalue path at $\omega$ space for various values of $u_{*}$, inviscid model, "one" profile, temporal case תנועת הערכים העצמיים במישור $\omega$ עבור המודל הבלתי צמיג ופרופיל "אחד״", התפתחות בזמן


Figure A.3: Normalized phase velocity vs. wavelength for various values of $u_{*}$, inviscid model, "one" profile, spatial case מהירות הפזה המנורמלת כתלות באורך הגל עבור המודל הבלתי צמיג ופרופיל ״אחד״, התפתחות במרחב


Figure A.4: Eigenvalue path at $k$ space for various values of $u_{*}$, inviscid model, "one" profile, spatial case

תנועת הערכים העצמיים במישור k עבור המודל הבלתי צמיג ופרופיל ״אחד״, התפתחות במרחב


Figure A.5: Max growth vs. friction velocity $u_{*}$, inviscid model "one" profile הגידול המקסימלי האפשרי כתלות במהירות החיכוך $u^{*}$ עבור המודל הבלתי צמיג


Figure A.6: Wavelength/waveperiod at max growth vs. friction velocity $u_{*}$, inviscid model, "one" profile

אורך הגל / זמן המחזור שבו מתקבל מקסימום הגידול כתלות במהירות החיכוך


Figure A.7: Friction velocity over calculated phase velocity at max growth vs. friction velocity $u_{*}$, inviscid model, "one" profile מהירות החיכוך מנורמלת במהירות הגל המחושבת בנקודת הגידול המקסימלי כתלות במהירות החיכוך $u^{*}$ עבור המודל הבלתי צמיג ופרופיל "אחד"


Figure A.8: Ratio between spatial and temporal growth vs. wavelength of the reference problem for various values of $u_{*}$, inviscid model, "one" profile יחס קצבי הגידול במרחב ובזמן כתלות באורך הגל של בעיית היחוס עבור ערכי שונים, מודל בלתי צמיג, פרופיל אחד


Figure A.9: Ratio between spatial and temporal normalized growth vs. wavelength of the reference problem for various values of $u_{*}$, inviscid model, "one" profile

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יחס קצבי הגידול המנורמלים במרחב ובזמן כתלות באורך הגל של בעיית היחוס עבור ערכי \({ }^{*}\) שונים, מודל בלתי צמיג, פרופיל "אחד״
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Figure A.10: Ratio between spatial and temporal phase velocity vs. wavelength of the reference problem for various values of $u_{*}$, inviscid model, "one" profile יחס מהירויות הגל במרחב ובזמן כתלות באורך הגל של בעיית היחוס עבור ערכי שונים, מודל בלתי צמיג פרופיל "אחד"


Figure A.11: Normalized phase velocity vs. wavelength for various values of $u_{*}$, inviscid model, "numerical" profile, temporal case

[^1]

Figure A.12: Waveperiod vs. wavelength for various values of $u_{*}$, inviscid model, "numerical" profile, temporal case זמן המחזור כתלות באורך הגל עבור המודל הבלתי צמיג ופרופיל ״נומר״, התפתחות בזמו


Figure A.13: Eigenvalue path at $\omega$ space for various values of $u_{*}$, inviscid model,
"numerical" profile, temporal case

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תנועת הערכים העצמיים במישור \omega עבור המודל הבלתי צמיג ופרופיל "נומרו" ,
    התפתחות בזמן
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Figure A.14: Normalized phase velocity vs. wavelength for various values of $u_{*}$, inviscid model, "numerical" profile, spatial case מהירות הפזה המנורמלת כתלות באורך הגל עבור המודל הבלתי צמיג ופרופיל "נומרי", התפתחות במרחב


Figure A.15: Wavelength velocity vs. wavelength for various values of $u_{*}$, inviscid model, "numerical" profile, spatial case אורך הגל כתלות באורך הגל עבור המודל הבלתי צמיג ופרופיל ״נומר״״, התפתחות במרחב


Figure A.16: Eigenvalue path at $k$ space for various values of $u_{*}$, inviscid model, "numerical" profile, spatial case

תנועת הערכים העצמיים במישור k עבור המודל הבלתי צמיג ופרופיל ״נומר״י, התפתחות במרחב


Figure A.17: Ratio between spatial and temporal growth vs. wavelength of the reference problem for various values of $u_{*}$, inviscid model, "numerical" profile יחס קצבי הגידול במרחב ובזמן כתלות באורך הגל של בעיית היחוס עבור ערכי


Figure A.18: Ratio between spatial and temporal phase velocity vs. wavelength of the reference problem for various values of $u_{*}$, inviscid model, "numerical" profile יחס מהירויות הגל במרחב ובזמן כתלות באורך הגל של בעיית היחוס עבור ערכי שונים, מודל בלתי צמיג, פרופיל "נומרי"


Figure A.19: Normalized phase velocity vs. wavelength for various values of $u_{*}$, viscous model, lin-log profile, temporal case

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מהירות הפזה המנורמלת כתלות באורך הגל עבור המודל הצמיג ופרופיל
    ליניארי־לוגריתמי , התפתחות בזמן
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Figure A.20: Waveperiod vs. wavelength for various values of $u_{*}$, viscous model,
lin-log profile, temporal case

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זמן המחזור כתלות באורך הגל עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי ,
``` התפתחות בזמן


Figure A.21: Eigenvalue path for various values of \(u_{*}\), viscous model, lin-log profile,
temporal case
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תנועת הערכים העצמיים במישור w עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי,
התפתחות בזמ

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Figure A.22: Maximum growth rate vs. friction velocity \(u_{*}\), viscous model, lin-log profile, temporal case קצב גידול מקסימלי כתלות במהירות החיכוך u* עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי , התפתחות בזמן


Figure A.23: Wavelength at maximum growth rate vs. friction velocity \(u_{*}\), viscous model, lin-log profile, temporal case

אורך הגל בו מתקבל גידול מקסימלי כתלות במהירות החיכוך u עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי, התפתחות בזמן


Figure A.24: Neutral wavelength vs. friction velocity \(u_{*}\), viscous model, lin-log profile, temporal case אורך הגל הנטראלי כתלות במהירות החיכוך ליניארי־לוגריתמי , התפתחות בזמן


Figure A.25: Calculated phase velocity over friction velocity at maximum growth rate vs. friction velocity \(u_{*}\), viscous model, lin-log profile, temporal case מהירות הפזה בנקודת מקסימום הגידול מנורמלת במהירות החיכוך כתלות במהירות החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי , התפתחות בזמן


Figure A.26: Calculated phase velocity over friction velocity at the neutral wave vs. friction velocity \(u_{*}\), viscous model, lin-log profile, temporal case מהירות הפזה בנקודת הגל הניטראלי מנורמלת במהירות החיכוך כתלות במהירות החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי , התפתחות בזמן


Figure A.27: Normalized phase velocity vs. wavelength for various values of \(u_{*}\), viscous model, "lin-log" profile, spatial case
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מהירות הפזה המנורמלת כזמן המחזור עבור המודל הצמיג ופרופיל ליניארי לוגריתמי ,
התפתחות במרחב

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Figure A.28: wavelength vs. Waveperiod for various values of \(u_{*}\), viscous model,
"lin-log" profile, spatial case אורך הגל כתלות בזמן המחזור עבור המודל הצמיג ופרופיל ליניארי לוגריתמי ,

התפתחות במרחב


Figure A.29: Eigenvalue path for various values of \(u_{*}\), viscous model, "lin-log" profile, spatial case תנועת הערכים העצמיים במישור k עבור המודל הצמיג ופרופיל ליניארי לוגריתמי, התפתחות במרחב


Figure A.30: Maximum growth rate vs. friction velocity \(u_{*}\), viscous model, "lin-log" profile, spatial case

קצב גידול מקסימלי כתלות במהירות החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל ליניארי לוגריתמי , התפתחות במרחב


Figure A.31: Waveperiod at maximum growth rate vs. friction velocity \(u_{*}\), viscous model, "lin-log" profile, spatial case

זמן המחזור בו מתקבל גידול מקסימלי כתלות במהירות החיכוך \({ }^{\text {ה }}\) עבור המודל הצמיג ופרופיל ליניארי לוגריתמי, התפתחות במרחב


Figure A.32: Neutral waveperiod vs. friction velocity \(u_{*}\), viscous model, "lin-log" profile, spatial case

זמן המחזור הקטן ביותר שנוצר כתלות במהירות החיכוך ליניארי לוגריתמי , התפתחות במרחב


Figure A.33: Calculated phase velocity over friction velocity at maximum growth rate vs. friction velocity \(u_{*}\), viscous model, "lin-log" profile, spatial case מהירות הפזה בנקודת מקסימום הגידול מנורמלת במהירות החיכוך כתלות במהירות החיכוך u עבור המודל הצמיג ופרופיל ליניארי לוגריתמי , התפתחות במרחב


Figure A.34: Calculated phase velocity over friction velocity at the neutral wave vs. friction velocity \(u_{*}\), viscous model, "lin-log" profile, spatial case מהירות הפזה בנקודת הגל הניטראלי מנורמלת במהירות החיכוך כתלות במהירות החיכוך u עבור המודל הצמיג ופרופיל ליניארי לוגריתמי, התפתחות במרחב


Figure A.35: Normalized phase velocity vs. wavelength for various values of \(u_{*}\), viscous model, "numerical" profile, temporal case
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מהירות הפזה המנורמלת כתלות באורך הגל עבור המודל הצמיג ופרופיל "נומר״",
התפתחות בזמן

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Figure A.36: Waveperiod vs. wavelength for various values of \(u_{*}\), viscous model, "numerical" profile, temporal case זמן המחזור כתלות באורך הגל עבור המודל הצמיג ופרופיל ״נומרֵ״, התפתחות בזמן


Figure A.37: Eigenvalue path for various values of \(u_{*}\), viscous model, "numerical" profile, temporal case


Figure A.38: Maximum growth rate vs. friction velocity \(u_{*}\), viscous model,
"numerical" profile, temporal case קצב גידול מקסימלי כתלות במהירות החיכוך u עבור המודל הצמיג ופרופיל "נומרי", התפתחות בזמן


Figure A.39: Wavelength at maximum growth rate vs. friction velocity \(u_{*}\), viscous model, "numerical" profile, temporal case אורך הגל בו מתקבל גידול מקסימלי כתלות במהירות החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל ״נומר״״, התפתחות בזמן


Figure A.40: Neutral wavelength vs. friction velocity \(u_{*}\), viscous model, "numerical" profile, temporal case אורך הגל הנטראלי כתלות במהירות החיכוך \(u_{\text {ה עבור המודל הצמיג ופרופיל "נומר״", }, ~}^{\text {ערי }}\) התפתחות בזמן


Figure A.41: Calculated phase velocity over friction velocity at maximum growth rate vs. friction velocity \(u_{*}\), viscous model, "numerical" profile, temporal case מהירות הפזה בנקודת מקסימום הגידול מנורמלת במהירות החיכוך כתלות במהירות החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל ״נומרי", התפתחות בזמן


Figure A.42: Calculated phase velocity over friction velocity at the neutral wave vs.
friction velocity \(u_{*}\), viscous model, "numerical" profile, temporal case מהירות הפזה בנקודת הגל הניטראלי מנורמלת במהירות החיכוך כתלות במהירות החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל ״נומרֵ״, , התפתחות בזמן


Figure A.43: Temporal growth rate vs. friction velocity \(u_{*,}\), viscous model, "numerical" profile, temporal case


Figure A.44: Phase velocity vs. friction velocity \(u_{*}\), viscous model, "numerical" profile, temporal case מהירות הפזה כתלות במהירות החיכוך \(u^{*}\) עבור המודל הצמיג ופרופיל ״נומרי״, התפתחות בזמן


Figure A.45: Normalized phase velocity vs. friction velocity \(u_{*}\), viscous model, "numerical" profile, temporal case

מהירות הפזה המנורמלת כתלות במהירות החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל "נומר", התפתחות בזמן


Figure A.46: Normalized phase velocity vs. wavelength for various values of \(u_{*}\), viscous model, "numerical" profile, spatial case
מהירות הפזה המנורמלת כזמן המחזור עבור המודל הצמיג ופרופיל ״נומר״״, , התפתחות במרחב


Figure A.47: Wavelength vs. waveperiod for various values of \(u_{*}\) viscous model,
"numerical" profile, spatial case
אורך הגל כתלות בזמן המחזור עבור המודל הצמיג ופרופיל ״נומר״״, , התפתחות במרחב


Figure A.48: Eigenvalue path for various values of \(u_{*}\), viscous model, "numerical" profile, spatial case

תנועת הערכים העצמיים במישור k עבור המודל הצמיג ופרופיל ״נומרי״, התפתחות
במרחב


Figure A.49: Maximum growth rate vs. friction velocity \(u_{*}\), viscous model, "numerical" profile, spatial case קצב גידול מקסימלי כתלות במהירות החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל ״נומרז’, התפתחות במרחב


Figure A.50: Waveperiod at maximum growth rate vs. friction velocity \(u_{*}\), viscous model, "numerical" profile, spatial case

זמן המחזור בו מתקבל גידול מקסימלי כתלות במהירות החיכוך \({ }^{\text {ג עבור המודל הצמיג }}\) ופרופיל ״נומרי״, התפתחות במרחב


Figure A.51: Neutral waveperiod vs. friction velocity \(u_{*}\), viscous model, "numerical" profile, spatial case

זמן המחזור הקטן ביותר שנוצר כתלות במהירות החיכוך * עבור המודל הצמיג ופרופיל "נומרז", , התפתחות במרחב


Figure A.52: Calculated phase velocity over friction velocity at maximum growth rate vs. friction velocity \(u_{*}\), viscous model, "numerical" profile, spatial case מהירות הפזה בנקודת מקסימום הגידול מנורמלת במהירות החיכוך כתלות במהירות החיכוך u עבור המודל הצמיג ופרופיל ״נומר״״, התפתחות במרחב


Figure A.53: Calculated phase velocity over friction velocity at the neutral wave vs.
friction velocity \(u_{*}\), viscous model, "numerical" profile, spatial case
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מהירות הפזה בנקודת הגל הניטראלי מנורמלת במהירות החיכוך כתלות במהירות

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    החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל ״נומרי", התפתחות במרחב


Figure A.54: Ratio between spatial and temporal phase velocity vs. wavelength of the reference problem for various values of \(u_{*}\), viscous model, "numerical" profile יחס מהירויות הפזה במרחב ובזמן כתלות באורך הגל של בעיית היחוס עבור ערכי שונים, המודל הצמיג, פרופיל ״נומרֵ״


Figure A.55: Waveperiod vs. wavelength comparison between spatial and temporal cases for various values of \(u_{*}\) viscous model, "numerical" profile

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    שונים, המודל הצמיג, פרופיל "נומרי"
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\section*{תקציר}








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שבירת הגלים.



 השולטת בבעיה, משוואת אור־סומרפלד למודל הצמיג ומשוואת רול למודל הבלתי צמי







צורך להשתמש בשיטות נומריות. השיטות הנומריות נבחרו באופן פרטני עבור המודל הצמיג והמודל הבלתי צמיג על מנת לספק את דרישות הדיוק והיציבות. הפתרון הנומרי של המודל הבלתי צמיג פשוט יותר והוא עושה שימוש בשיטת הירייה ופותרן סטנדרטי לבעיית התחלה. הפתרון הנומרי של הבעיה הצמיגה מורכב יותר כיוון שלא ניתן לבצע

אינטגרציה למשוואה, בפתרון זה נעשה שימוש בשיטת הקולוקציות של צ׳בישב. בגישה בה אנו נוקטים לפתרון בעיית היציבות אחד האיברים המשמעותיים ביותר המופיע בפיתוח הינו \(U(z\) אשר מסמל את פרופיל מהירות הרקע. בגישה זו, הקרויה לפעמים הגישה של מיילס (14 (14) ] או הגישה הקווזי־למינארית, הזרימה הטורבולנטית נלקחת בחשבון דרך זרימת הרקע. ההנחה שזרימה מעל גלים הינה טורבולנטית מקובלת על רבים. על פי גשת מיילס מזניחים את התנודות הטורבולנטיות ומתחשבים רק בפרופיל המהירות הממוצעת, בעוד שהרכיבים הגליים של הזרימה הינם למינריים לחלוטין. הבעייתיות במודל זה הינה הרגישות לבחירת פרופיל המהירות הממוצעת. בעבודה זו החישובים מתבצעים עם שלוש גרסאות שונות של פרופיל מהירות. שניים מתוכם הינם פרופילי מהירות מוכרים והשלישי הינו גרסה חדשה לפרופיל המהירות אשר מתבססת על תיאוריית אורך הערבוב של פרנטל. נושא בחירת פרופיל המהירות הינו משמעותי כפי שכבר צוין, במסגרת זו החלק המשמעותי ביותר הינו פרופיל המהירות בשכבה הקרובה לפני המים מנגנון יצירת הגלים במודל הבלתי צמיג המבוסס על משוואת ריילי הינו פשוט יותר ומופיע בבעיות רבות הקשורות ליציבות, בעיקר בנושא מעבר מזרימה למינרית לזרימה טורבולנטית. במודל זה תנאי הכרחי להיווצרות הגלים (אי־יציבות) הינו 0> 0 ( \(U^{\prime \prime}\left(z_{c}\right.\) כאשר הנקודה הקריטית \(z_{c}\) היא הנקודה שבה \(U\left(z_{c}\right)=c_{r}\) והגידול הינו יחסי לגודל העקמומיות בנקודה זו. כאשר cr היא מהירות הגל. במודל הצמיג (משוואת אור־סומרפלד) מנגנון הגי דול (אי־יציבות) הינו מורכב יותר וקשה להגדיר בצורה פשוטה את התנאים לגידול, או

לאתר במפורש את הגורמים השולטים בעוצמת הגידול. חישוב הערכים העצמיים מתבצע עבור מקרה מסוים בכל פעם, המקרה מוגדר ע״י עוצמת הרוח וערכים של בעיית הייחוס: אורך הגל עבור הבעיה בזמן וזמן המחזור עבור הבעיה במרחב. תוצרי החישוב הם הערכים העצמיים והפונקציות העצמיות. מהערכים העצמיים ניתן לדעת את קצב הגידול/דעיכה של הגלים ומהפונקציות העצמיות ניתן לחשב את כל שדה הזרימה (מהירות ולחץ). החישובים בעבודה זו בוצעו עבור תרחישים רבים הן למצב של אי־יציבות במרחב והן למצב של אי־יציבות בזמן. ההרחבה המשמעותית הינה למודל הצמיג בו רוב התוצאות שפורסמו בעבר היו למקרה של אי־יציבות בזמן ועבור טווח קטן של ערכי עוצמת הרוח ואורכי גל. הסיבה המשוערת לכך הינה מגבלות נומריות של חישוב הבעיה. אנו הבחנו באותה מגבלה נומרית אשר מופיעה באורכי גל ארוכים

ובעוצמות רוח גדולות. הרחבת טווח החישוב למודל זה אפשרה להבחין בקיומו של פתרון נוסף שאינו יציב, כלומר עבור מקרה מסוים יכולים להתקיים שני גלים שאינם יציבים. התנאים המדויקים לקיום מצב זה עדיין אינם ברורים אך ברור כי תנאי סף לקיום המצב הינם: קיום זרימת גזירה במים, עוצמת רוח מעל סף מסוים. קצב הגידול המתקבל בשני ענפי הפתרון המדוברים הינו שונה אך השוני המהותי יותר הינו בערכי מהירות הפאזה המתקבלת. המשמעות יכולה להיות שינוי של משטר הגלים כאשר ישנה דומיננטיות של פתרון זה או אחר. פרופיל המהירות החדש שמוצע פתח את האפשרות לביצוע השוואה "הוגנת״ בין המודל הצמיג למודל הבלתי צמיג. השוואה זו חשובה כיוון שבמקרים רבים מקובל להשתמש בקירוב הבלתי צמיג. השוואה מקיפה בין שני המודלים המבוססת על השוואת הערכים העצמיים, הפונקציות העצמיות ומשטח התנאי הדינאמי (ראה פרק תוצאות) מוצגת בעבודה. מהשוואה זו עולה כי בעוצמות רוח נמוכות יחסית הפתרונות קרובים במובן מסוים אך מבחינת ערכי הגידול ישנו הבדל משמעותי הגדול מ־100\% בין המודלים. בעוצמות רוח חזקות נראה כי המודלים מתרחקים זה מזה וההבדלים הינם משמעותיים ביותר. השוואה זו מעלה שאלות לגבי תוקף השימוש בקירוב הבלתי צמיג

יצירת גלי מים על ידי הרוח

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המחקר נעשה בהדרכת פרופ. מיכאל שטיאסני ופרופ. יהודה עגנון
הפקולטה להנדסה אזרחית וסביבתית

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\section*{הכרת תודה}

ברצוני להודות למנחי פרופ׳ מיכאל שטיאסני ופרופ' יהודה עגנון על הנחייתם המסורה, על שנתנו לי מנסיונם על שהיו לי אוזן קשבת, תמכו ועודדו לאורך כל הדרך.

למשפחתי

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}

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הוגש לסנט הטכניון מכון טכנולוגי לישראל

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23.א׳ אורך הגל בו מתקבל גידול מקסימלי כתלות במהירות החיכוך u עבור \(^{\text {עב }}\) המודל הצמיג ופרופיל ליניארי־לוגריתמי, התפתחות בזמן 24. א׳ אורך הגל הנטראלי כתלות במהירות החיכוך u* עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי , התפתחות בזמן 25.א׳ מהירות הפזה בנקודת מקסימום הגידול מנורמלת במהירות החיכוך כתלות במהירות החיכוך \(u^{*}\) עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי , התפתחות בזמן
26.א׳ מהירות הפזה בנקודת הגל הניטראלי מנורמלת במהירות החיכוך כתלות במהירות החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל ליניארי־לוגריתמי , התפתחות בזמן
27. א׳ מהירות הפזה המנורמלת כזמן המחזור עבור המודל הצמיג ופרופיל ליניארי לוגריתמי , התפתחות במרחב 28. א׳ אורך הגל כתלות בזמן המחזור עבור המודל הצמיג ופרופיל ליניארי לוגריתמי התפתחות במרחב
29. א׳ תנועת הערכים העצמיים במישור k עבור המודל הצמיג ופרופיל ליניארי לוגריתמי, התפתחות במרחב
30.א׳ קצב גידול מקסימלי כתלות במהירות החיכוך u \(_{\text {ה עבור המודל הצמיג }}\) ופרופיל ליניארי לוגריתמי , התפתחות במרחב
31.א׳ זמן המחזור בו מתקבל גידול מקסימלי כתלות במהירות החיכוך u עבור המודל הצמיג ופרופיל ליניארי לוגריתמי, התפתחות במרחב
32.א׳ זמן המחזור הקטן ביותר שנוצר כתלות במהירות החיכוך u עבור המודל הצמיג ופרופיל ליניארי לוגריתמי , התפתחות במרחב
33.א׳ מהירות הפזה בנקודת מקסימום הגידול מנורמלת במהירות החיכוך כתלות במהירות החיכוך \({ }^{*}\) עבור המודל הצמיג ופרופיל ליניארי לוגריתמי , התפתחות במרחב
34.א׳ מהירות הפזה בנקודת הגל הניטראלי מנורמלת במהירות החיכוך כתלות במהירות החיכוך u* עבור המודל הצמיג ופרופיל ליניארי לוגריתמי , התפתחות במרחב
35.א׳ מהירות הפזה המנורמלת כתלות באורך הגל עבור המודל הצמיג ופרופיל "נומר״״, , התפתחות בזמן
36.א׳ זמן המחזור כתלות באורך הגל עבור המודל הצמיג ופרופיל "נומרֵ" , התפתחות בזמן
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\section*{הכרת תודה}```


[^0]:    מהירות הפזה המנורמלת כתלות באורך הגל עבור המודל הבלתי צמיג ופרופיל "אחד", התפתחות בזמן

[^1]:    מהירות הפזה המנורמלת כתלות באורך הגל עבור המודל הבלתי צמיג ופרופיל "נומר"״, התפתחות בזמן

