Statphys Exam, Feb 22, 2007

1. A lattice in one dimension has N cites and is at temperature T. At each cite there is an atom which can be in either of two energy states: $E_i = \pm\epsilon$. When L consecutive atoms are in the $+\epsilon$ state, we say that they form a cluster of length L (provided that the atoms adjacent to the ends of the cluster are in the state $-\epsilon$). In the limit $N \to \infty$,

a) Compute the probability $\mathcal{P}_L$ that a given cite belongs to a cluster of length L (don't forget to check that $\sum_{L=0}^{\infty} \mathcal{P}_L = 1$);

b) Calculate the mean length of a cluster $\langle L \rangle$ and determine its low- and high-temperature limits.

2. A cavity containing a gas of electrons has a small hole of area A through which electrons can escape. External electrodes are so arranged that voltage is V between inside and outside of the cavity. Assume that

i) a constant number density of electrons is maintained inside (for example, by thermionic emission);

ii) electrons are in thermal equilibrium with temperature T and chemical potential μ such that $kT \ll V - \mu$;

iii) electrons moving towards the hole escape if they have an energy greater than V.

Estimate the total current carried by escaping electrons.

3. A d-dimensional container is divided into two regions A and B by a fixed wall. The two regions contain identical Fermi gases of spin 1/2 particles which have a magnetic moment τ. In region A there is a magnetic field of strength H, but there is no field in region B. Initially, the entire system is at zero temperature, and the numbers of particles per unit volume are the same in both regions. If the wall is now removed, particles may flow from one region to the other. Determine the direction in which particles begin to flow, and how the answer depends on the space dimensionality d.

4. Consider the spin-1 Ising model on a cubic lattice in d dimensions, given by the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j - \Delta \sum_i S_i^2 - h \sum_i Si \ ,$$

where $S_i = 0, \pm 1$, $\sum_{<ij>}$ denote a sum over z nearest neighbor sites and $J, \Delta > 0$.

(a) Write down the equation for the magnetization $m = \langle S_i \rangle$ in the mean-field approximation.

(b) Calculate the transition line in the (T, Δ) plane (take $h = 0$) which separates the paramagnetic and the ferromagnetic phases. Here T is the temperature.

(c) Calculate the magnetization (for $h = 0$) in the ferromagnetic phase near the transition line, and show that to leading order $m \sim \sqrt{T_c - T}$, where T_c is the transition temperature.

(d) Show that the zero-field ($h = 0$) susceptibility χ in the paramagnetic phase is given by

$$\chi = \frac{1}{k_B T} \frac{1}{1 + \frac{1}{2} e^{-\beta\Delta} - \frac{Jz}{k_B T}} \ .$$

where $\beta = 1/k_B T$.

Answers

Problem 1.

a) Probabilities of any cite to have energies $\pm\epsilon$ are

$$\mathcal{P}_{\pm} = e^{\pm\beta\epsilon}(e^{\beta\epsilon} + e^{-\beta\epsilon})^{-1} \ .$$

The probability for a given cite to belong to an L-cluster is $\mathcal{P}_L = L\mathcal{P}_+^L\mathcal{P}_-^2$ for $L \geq 1$ since cites are independent and we also need two adjacent cites to have $-\epsilon$. The cluster of zero length corresponds to a cite having $-\epsilon$ so that $\mathcal{P}_L = \mathcal{P}_-$ for $L = 0$. We ignore the possibility that a given cite is within L of the ends of the lattice, it is legitimate at $N \to \infty$.

$$\sum_{L=0}^{\infty} \mathcal{P}_L = \mathcal{P}_- + \mathcal{P}_-^2 \sum_{L=1}^{\infty} L\mathcal{P}_+^L = \mathcal{P}_- + \mathcal{P}_-^2\mathcal{P}_+ \frac{\partial}{\partial\mathcal{P}_+} \sum_{L=1}^{\infty} \mathcal{P}_+^L = \mathcal{P}_- + \frac{\mathcal{P}_-^2\mathcal{P}_+}{(1-\mathcal{P}_+)^2} = \mathcal{P}_- + \mathcal{P}_+ = 1$$

b)

$$\langle L\rangle = \sum_{L=0}^{\infty} L\mathcal{P}_L = \mathcal{P}_-^2\mathcal{P}_+ \frac{\partial}{\partial\mathcal{P}_+} \sum_{L=1}^{\infty} L\mathcal{P}_+^L = \frac{\mathcal{P}_+(1+\mathcal{P}_+)}{\mathcal{P}_-} = e^{-2\beta\epsilon}\frac{e^{\beta\epsilon} + 2e^{-\beta\epsilon}}{e^{\beta\epsilon} + e^{-\beta\epsilon}} \ .$$

At $T = 0$ all cites are in the lower level and $\langle L\rangle = 0$. As $T \to \infty$, the probabilities $\mathcal{P}_+$ and $\mathcal{P}_-$ are equal and the mean length approaches its maximum $\langle L\rangle = 3/2$.

Problem 4.

a) $\mathcal{H}_{eff}(S) = -JmzS - \Delta S^2 - hS$, $S = 0, \pm 1$.

$$m = e^{\beta\Delta}\frac{e^{\beta(Jzm+h)} - e^{-\beta(Jzm+h)}}{1 + e^{\beta\Delta}\left[e^{\beta(Jzm+h)} + e^{-\beta(Jzm+h)}\right]} \ .$$

b) $h = 0$,

$$m \approx e^{\beta\Delta}\frac{2\beta Jzm + (\beta Jzm)^3/3}{1 + 2e^{\beta\Delta}[1 + (\beta Jzm)^2/2]} \ .$$

Transition line $\beta_c Jz = 1 + \frac{1}{2}e^{-\beta_c\Delta}$.

c)

$$m^2 = \frac{(\beta - \beta_c)Jz}{(\beta_c Jz)^2/2 - (\beta_c Jz)^3/6} \ .$$

d)

$$m \approx e^{\beta\Delta}\frac{2\beta Jzm + 2\beta h}{1 + 2e^{\beta\Delta}}\ ,\ m \approx 2\beta h(2 + e^{-\beta\Delta} - \beta Jz)^{-1}\ ,\ \chi = \partial m/\partial h \ .$$

Problem 3.

Particles flow from a region of higher chemical potential to a region of lower potential. We need to find in which region the chemical potential is higher, and we do this by considering the grand canonical expression for the number of particles per unit volume. In the presence of a magnetic field, the single-particle energy is $\varepsilon \pm \tau H$, where ε is the kinetic energy, depending on whether the magnetic moment is parallel or antiparallel to the field. The total number of particles is then given by

$$N = \int_0^\infty \mathrm{d}\varepsilon\, g(\varepsilon) \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu-\tau H)}+1} + \int_0^\infty \mathrm{d}\varepsilon\, g(\varepsilon) \frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu+\tau H)}+1}.$$

For non-relativistic particles in a d-dimensional volume V, the density of states is $g(\varepsilon) = \gamma V \varepsilon^{d/2-1}$, where γ is a constant. At $T = 0$, the Fermi distribution function is

$$\lim_{\beta\to\infty} \left(\frac{1}{\mathrm{e}^{\beta(\varepsilon-\mu\pm\tau H)}+1} \right) = \theta(\mu \mp \tau H - \varepsilon)$$

where $\theta(\cdot)$ is the step function, so the integrals are easily evaluated with the result

$$\frac{N}{V} = \frac{2\gamma}{d}[(\mu+\tau H)^{d/2} + (\mu - \tau H)^{d/2}].$$

At the moment that the wall is removed, N/V is the same in regions A and B; so (with $H = 0$ in region B) we have

$$(\mu_\mathrm{A} + \tau H)^{d/2} + (\mu_\mathrm{A} - \tau H)^{d/2} = 2\mu_\mathrm{B}^{d/2}.$$

For small fields, we can make use of the Taylor expansion

$$(1 \pm x)^{d/2} = 1 \pm \frac{d}{2}x + \frac{d}{4}\left(\frac{d}{2} - 1\right)x^2 + \cdots$$

to obtain

$$\left(\frac{\mu_\mathrm{B}}{\mu_\mathrm{A}}\right)^{d/2} = 1 + \frac{d(d-2)}{8}\left(\frac{\tau H}{\mu_\mathrm{A}}\right)^2 + \cdots.$$

We see that, for $d = 2$, the chemical potentials are equal, so there is no flow of particles. For $d > 2$, we have $\mu_\mathrm{B} > \mu_\mathrm{A}$ and so particles flow towards the magnetic field in region A while, for $d < 2$, the opposite is true. We can prove that the same result holds for any magnetic field strength as follows. For compactness, we write $\lambda = \tau H$. Since our basic equation $(\mu_\mathrm{A} + \lambda)^{d/2} + (\mu_\mathrm{A} - \lambda)^{d/2} = 2\mu_\mathrm{B}^{d/2}$ is unchanged if we change λ to $-\lambda$, we can take $\lambda > 0$ without loss of generality. Bearing in mind that μ_B is fixed, we calculate $\mathrm{d}\mu_\mathrm{A}/\mathrm{d}\lambda$ as

$$\frac{\mathrm{d}\mu_\mathrm{A}}{\mathrm{d}\lambda} = \frac{(\mu_\mathrm{A} - \lambda)^{d/2-1} - (\mu_\mathrm{A} + \lambda)^{d/2-1}}{(\mu_\mathrm{A} - \lambda)^{d/2-1} + (\mu_\mathrm{A} + \lambda)^{d/2-1}}.$$

Since $\mu_\mathrm{A} + \lambda > \mu_\mathrm{A} - \lambda$, we have $(\mu_\mathrm{A} + \lambda)^{d/2-1} > (\mu_\mathrm{A} - \lambda)^{d/2-1}$ if $d > 2$ and vice versa. Therefore, if $d > 2$, then $\mathrm{d}\mu_\mathrm{A}/\mathrm{d}\lambda$ is negative and, as the field is increased, μ_A decreases from its zero-field value μ_B and is always smaller than μ_B. Conversely, if $d < 2$, then μ_A is always greater than μ_B. For $d = 2$, we have $\mu_\mathrm{A} = \mu_\mathrm{B}$ independent of the field.