Lecture 1. Optical Forces.

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Contents

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I. LIGHT PRESSURE ON A DIELECTRIC HALF-SPACE

We aim to find the pressure of light normally indicent on a dielectric half-space.

A. Naive answer

It is easy to obtain the answer in the limit $n \gg 1$ where $|r| \to 1$ and light is fully reflected. In this case the pressure is just given by the Newton's law, $p = dP/dt$, where P is the change of momentum of light when reflected from the surface of unit area. To find the momentum we remember that the incident light has energy density $(E^2 + H^2)/(8\pi)$ i.e. $E_0^2/2\pi$ on average. Here we assume that the time dependence is $\boldsymbol{E}(t) = \boldsymbol{E}_0 \mathrm{e}^{-\mathrm{i}\omega t} + \boldsymbol{E}_0^* \mathrm{e}^{\mathrm{i}\omega t}$ and take into account that the time average of $\overline{E(t)^2}$ is $2E_0^2$. Since for light the dispersion relation between momentum P and energy $\mathcal{E}$ for relativistic particles is $\mathcal{E} = cP$, the momentum density is $P = E_0^2/2\pi c$. The momentum going through unit area per unit time is then given by $cP = E_0^2/(2\pi)$. We also take into account that outgoing wave carries the same momentum but in the opposite direction. Hence, the pressure of light is

$$p = \frac{E_0^2}{\pi} \equiv \frac{2J}{c}, \quad \text{where } J = \frac{cE_0^2}{2\pi} \text{ is the light intensity, } \left[\frac{\mathrm{erg}}{\mathrm{cm}^2 \cdot \mathrm{s}}\right]. \tag{1}$$

Let us consider a grain of dust of the size 1 μm $\times$ 1 μm $\times$ 1 μm illuminated by a tightly focused laser pointer of the power 1 mW focused to the area of $A = 1\ \mu\mathrm{m}^2$. We find $J = 10^9$ J/m^2/s, $p \sim$ 10 N/m^2 and force $f_{\mathrm{rad}} = pA = 10$ pN. The pressure is weaker than atmospheric pressure by 5 orders of magnitude! (this was a problem for first experiments of Lebedew and of Nicholls and Hull). On the other hand, the volume is about 10^{-12} cm^3 and for the density of 1 g/cm^3 we find the mass of 10^{-15} kg. Hence, the gravitation force mg is much smaller than the light pressure force.

Now we try to generalize Eq. (1) for the medium with arbitrary complex refractive index n.

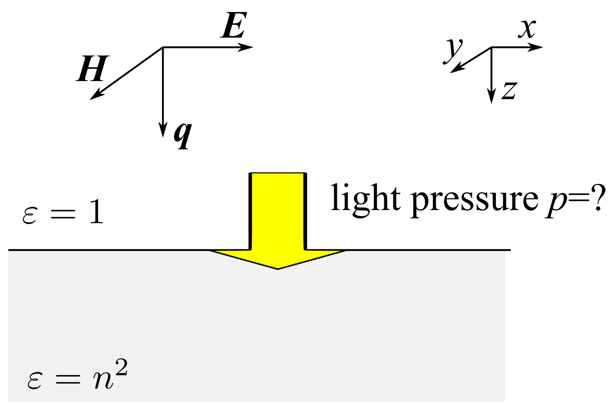

FIG. 1 Schematic illustration of a plane wave incident on a dielectric surface.

B. General formalism

We consider the geometry$\boldsymbol{E} \parallel x$, $\boldsymbol{H} \parallel z$, see Fig. 1. Maxwell's equations read:

$$[\mathrm{rot}\,\boldsymbol{E}]_y \equiv \frac{\partial E_x}{\partial z} = -\frac{1}{c}\frac{\partial H_y}{\partial t} \tag{2}$$

$$[\mathrm{rot}\,\boldsymbol{H}]_x \equiv -\frac{\partial H_y}{\partial z} = \frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{4\pi}{c} j_x \tag{3}$$

$$\frac{1}{c}\int \mathrm{d}z j_x H_y + \frac{1}{4\pi}\int \mathrm{d}z H_y \frac{\partial H_y}{\partial z} + \frac{1}{4\pi c}\int \mathrm{d}z \frac{\partial E_x}{\partial t} H_y = 0$$

$$H_y \frac{\partial H_y}{\partial z} = \frac{1}{2}\frac{\partial H_y^2}{\partial z}, \quad \frac{\partial E_x}{\partial t} H_y = \frac{\partial E_x H_y}{\partial t} - \frac{\partial H_y}{\partial t} E_x = \frac{\partial E_x H_y}{\partial t} + \frac{\partial E_x}{\partial z} E_x = \frac{\partial E_x H_y}{\partial t} + \frac{1}{2}\frac{\partial E_x^2}{\partial z}$$

The momentum conservation law reads:

$$\boxed{\int \mathrm{d}z \left[\frac{\partial G_z}{\partial t} - \frac{\partial \sigma_{zz}}{\partial z}\right] + p_z = 0,} \tag{4}$$

where

$$p_z = \int \mathrm{d}z j_x H_y \tag{5}$$

is the pressure (force/cm^2),

$$G_z = \frac{E_x H_y}{4\pi c} = \frac{1}{4\pi c}[\boldsymbol{E} \times \boldsymbol{H}]_z \tag{6}$$

is the light momentum density, and

$$\sigma_{zz} = -\frac{E_x^2 + H_y^2}{8\pi} \equiv \frac{1}{4\pi}\left[E_\alpha E_\beta + H_\alpha H_\beta - \frac{\delta_{\alpha\beta}}{2}(E^2 + H^2)\right]\Big|_{\alpha=\beta=z} \tag{7}$$

is the *Maxwell stress tensor* (in the Abraham formulation, see discussion in Sec. II.C). In the stationary case, e.g. for fields that harmonically change in time, the term $\partial G_z/\partial t$ is zero.

C. Application to light reflected from a dielectric half-space

The electromagnetic field is given by

$$E_x(z) = \begin{cases} E_0(\mathrm{e}^{\mathrm{i}qz} + r\mathrm{e}^{-\mathrm{i}qz}), & z < 0 \\ tE_0\mathrm{e}^{\mathrm{i}qnz}, & z > 0\,, \end{cases} \quad , \quad H_y(z) = \begin{cases} E_0(\mathrm{e}^{\mathrm{i}qz} - r\mathrm{e}^{-\mathrm{i}qz}), & z < 0 \\ tnE_0\mathrm{e}^{\mathrm{i}qnz}, & z > 0\,, \end{cases} \tag{8}$$

Here $t = 1 + r, r = (1 - n)/(1 + n)$, the half-space permittivity is $\varepsilon = n^2$ and $n = n' + \mathrm{i}n''$ is the complex refractive index. This derivation loosely follows a more general quantum approach in (Loudon, 2002).

$$p_z = \frac{1}{c}\int_0^\infty \mathrm{d}z j_x B_y\,. \tag{9}$$

$$j_x(t) = j_x\mathrm{e}^{-\mathrm{i}\omega t} + j_x^*\mathrm{e}^{\mathrm{i}\omega t}, \quad B_y(t) = B_y\mathrm{e}^{-\mathrm{i}\omega t} + B_y^*\mathrm{e}^{\mathrm{i}\omega t}, \quad \overline{j_x(t)B_y(t)} = 2\,\mathrm{Re}\, j_x B_y^*$$

where the overlain stands for the time average. The current can be expressed via the dielectric polarization $P = (\varepsilon - 1)E/(4\pi)$.

$$j_x = \frac{\mathrm{d}P_x}{\mathrm{d}t} = -\mathrm{i}\omega\frac{\varepsilon - 1}{4\pi}E_x \tag{10}$$

We need to express real pressure via the complex amplitudes, Using Eq. (8) we find

$$\begin{aligned} \frac{1}{c}\int_0^\infty \mathrm{d}z j_x B_y = \frac{2\omega}{4\pi c}|t|^2|E_0|^2\,\mathrm{Re}\,[-\mathrm{i}(\varepsilon - 1)n^*]\int_0^\infty \mathrm{d}z\mathrm{e}^{-2\omega n''/c} = \frac{|E_0|^2}{\pi|n+1|^2}\,\mathrm{Im}\,\frac{[(n^2-1)n^*]}{n''} \\ = \frac{|E_0|^2}{\pi|n+1|^2}\,\mathrm{Im}\,\frac{|n|^2 n - n^*}{n''} = \frac{|E_0|^2(|n^2|+1)}{\pi|n+1|^2}\,. \end{aligned} \tag{11}$$

Introducing the light intensity $J = cE_0^2/(2\pi)$ we rewrite the answer as

$$p = \frac{2J}{c}\frac{(|n^2|+1)}{\pi|n+1|^2}\,. \tag{12}$$

In case when $n \gg 1$ and reflection is close to 100%, Eq. (12) reduces to Eq. (1), $p = 2J/c$.

D. Approach via Maxwell stress tensor

Alternatively, we could obtain the same answer just by evaluating the Maxwell stress tensor at $z \to -0$. In this case

$$E_x(z \to -0) = E_0(1 + r), \quad H_y(z \to -0) = E_0(1 - r) \tag{13}$$

$$[-\sigma_{zz}]_{z=-0} = 2\frac{E_x^2 + H_y^2}{8\pi} = \frac{E_0^2}{4\pi}(|1+r|^2 + |1-r|^2) = \frac{E_0^2}{2\pi}(1 + |r|^2) = \frac{E_0^2}{\pi}\frac{(|n^2|+1)}{|n+1|^2} \tag{14}$$

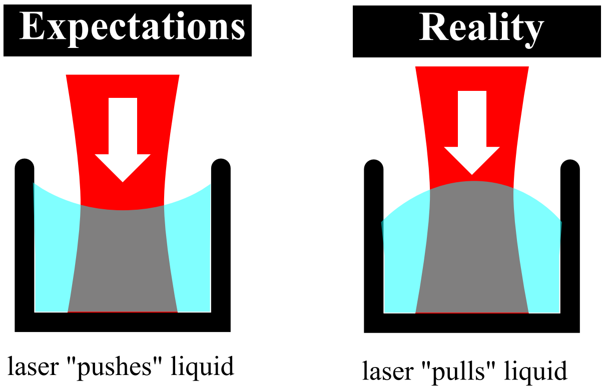

FIG. 2 Left: schematic illustration of experimental results from (Ashkin and Dziedzic, 1973): liquid is pulled by the laser beam instead of being pushed out by it, as could be naively expected from Eq. (1) for the pressure of light

E. Approach via change of light momentum

We could just calculate the pressure by assuming, that the momentum difference for photons near the dielectric is $J(1+|r|^2)/c$, where J/c is the flux of momentum for incoming wave and $J|r|^2/c$ is the (reversed) flux of momentum for the outgoing wave.

Disclaimer: this approach is in fact calculating exactly the same stress tensor contribution as in Sec. I.D and not the contribution $\partial G_z/\partial t$ that is zero for a monochromatic wave.

II. OPTICAL FORCES FOR A SMALL PARTICLE

In the previous section we can calculated the pressure of light on a dielectric half-space. It seems that light can only push objects forward.

In fact, some experiments done by Ashkin (Ashkin and Dziedzic, 1973), Nobel Prize in Physics (2018) have demonstrated just the opposite. The liquid was *attracted* by a tightly-focused laser beam, not repelled by it, see Fig. 2!

What is wrong with our derivation? The answer is that there is one more force, namely, a ponderomotive force! A dipole $\boldsymbol{p}$ in the electric field $\boldsymbol{E}$ has an energy $U = -\boldsymbol{p}\cdot\boldsymbol{E}$. Now let us assume that the dipole is induced by the same field, $\boldsymbol{p} = \alpha\boldsymbol{E}$ where α is the polarizability. Then we find $U = -\alpha E^2$ and the force

$$\boldsymbol{F} = -\nabla\boldsymbol{U} \propto \nabla(\alpha E^2), \tag{15}$$

so that for $\alpha > 0$ the particles are attracted to the maxima of electric field. This is what happens with the water in the glass under the laser beam illumination and why it raises upwards instead of been pressed down (Gordon, 1973). More details are given in Sec. II.C. However, Eq. (15) is not the final answer. It does not take into account that we are dealing with electromagnetic fields that dynamically change in time, rather than static fields. How should it be modified when we also consider the effect of the Lorentz force and the fact that the polarizability α is complex? We will now consider this in more detail.

A. Force, acting on a small electric dipole

Let us consider an atom, consisting of a heavy ion with large mass and a charge $-q$ and an electron with the charge $+q$. The force, acting on the atom as a whole, has two contributions: Lorentz force and electrostatic force acting upon the two charges:

$$\boldsymbol{F} = \frac{q}{c}\dot{\boldsymbol{r}} \times \boldsymbol{B} + q[\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{E}(0)] \approx \frac{q}{c}\dot{\boldsymbol{r}} \times \boldsymbol{B} + q(\boldsymbol{r}\cdot\boldsymbol{\nabla})\boldsymbol{E}(0) \equiv \frac{1}{c}\dot{\boldsymbol{p}} \times \boldsymbol{B}(t) + (\boldsymbol{p}\cdot\boldsymbol{\nabla})\boldsymbol{E}(t)\,. \tag{16}$$

Wave packet traveling through a gas of atoms

Let us consider a wave packet propagating along z direction and polarizing along y;

$$\boldsymbol{E}(z,t) = \boldsymbol{e}_x E(z-ct/n), \quad \boldsymbol{H} = \boldsymbol{e}_y n E(z-ct/n)$$

where n is the refracting index. In this case the force can be rewritten as

$$\boldsymbol{F}(z,t) = \frac{\alpha n \boldsymbol{e}_z}{2c}\frac{\partial}{\partial t}E(z-ct/n)^2\,. \tag{17}$$

where α is the dipole polarizability, $\boldsymbol{p} = \alpha\boldsymbol{E}$. In the lowest order in α we can write the refractive index of the gas as $n \approx 1 + 2\pi N\alpha$ where N is the concentration of atoms (we neglect local field corrections etc). As such, the volume density of force acting on atoms (Gordon, 1973) can be presented as

$$\boldsymbol{f}(z,t) = \frac{\partial}{\partial t}\boldsymbol{M}(z,t), \quad \boldsymbol{M}(z,t) = \frac{n-1}{4\pi c}E^2\boldsymbol{e}_z \approx \frac{n-1}{4\pi c}\boldsymbol{E}\times\boldsymbol{H}. \tag{18}$$

Here, $\boldsymbol{M}$ can be interpreted as *mechanicaI* momentum density of the atoms traveling through the gas with the pulse. We stress, that after the pulse passes a given point, no momentum is left in the gas ; total momentum is conserved at least in the linear order in $n-1$, see Fig. 3(b). However, when the pulse enters the gas through the boundary, some mechanical momentum is transferred the gas, see Fig. 3(a).

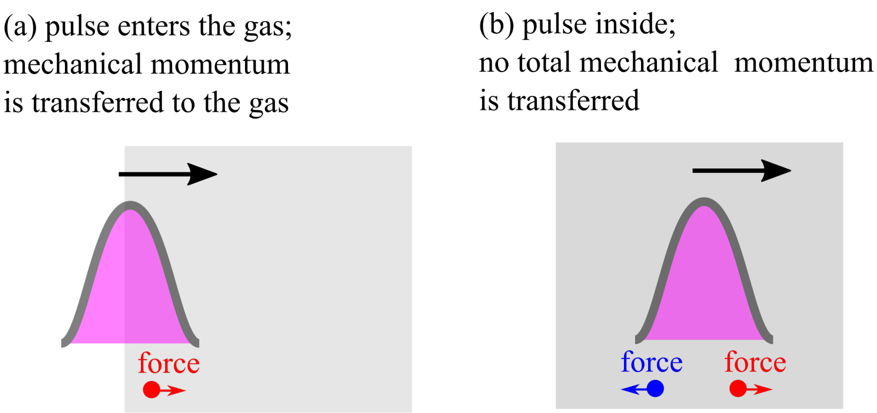

FIG. 3 Forces acting on particles when pulse (a) enters the gas and (b) propagates inside the gas. After Ref. (Gordon, 1973).

Monochromatic field acting upon a single particle

Next, we consider the harmonic fields and rewrite the time-averaged force Eq. (16) taken into account that

$$\boldsymbol{E}(t)=\boldsymbol{E}\mathrm{e}^{-\mathrm{i}\omega t}+c.c.,\quad \boldsymbol{B}(t)=\boldsymbol{B}\mathrm{e}^{-\mathrm{i}\omega t}+c.c.,\quad \boldsymbol{p}(t)=\alpha\boldsymbol{E}\mathrm{e}^{-\mathrm{i}\omega t}+c.c. \tag{19}$$

The time-averaged force then reads

$$\overline{\boldsymbol{F}(t)}\equiv\boldsymbol{F}_{\mathrm{rad}}+\boldsymbol{F}_{\mathrm{grad}}=\frac{2\omega}{c}\,\mathrm{Im}[\alpha\boldsymbol{E}\times\boldsymbol{B}^*]+2\,\mathrm{Re}[\alpha(\boldsymbol{E}\cdot\boldsymbol{\nabla})\boldsymbol{E}^*]\,. \tag{20}$$

The second, gradient, force, can lead to the trapping of the particle in the focus of strong electromagnetic wave. So the first contribution is proportional to the photon momentum and the second to the gradient of the photon power. The same two contributions exist in semiconductors, where an interaction of the electromagnetic wave with the charge carriers can lead to a steady current (Perel' and Pinskii, 1973). This is termed *photon drag effect*, see (Glazov and Ganichev, 2014) for a recent review of photon drag effect and other similar effects in graphene.

Important note. We dealing here with electromagnetic waves, rather than static fields. For static field we have $\Delta\boldsymbol{E}=0$ which means that $\Delta E^2=2\sum_\mu(\mathrm{grad}\,E_\mu)^2>0$ and stable maxima of electric field are impossible (Earnshaw theorem). However, this is not a problem for electromagnetic waves where tightly focused beams are possible.

For a plane electromagnetic wave propagating along the direction $\boldsymbol{n}$ in vacuum we have

$\boldsymbol{B} = \boldsymbol{n} \times \boldsymbol{E}$. In this case the force reads

$$\boldsymbol{F} \equiv \boldsymbol{F}_{\rm rad} = \frac{2\omega}{c}\,{\rm Im}\,\alpha |E|^2 \boldsymbol{n} = \sigma_{\rm ext}\frac{J}{c}\boldsymbol{n}\,, \tag{21}$$

where J is the light intensity and $\sigma_{\rm ext} = 4\pi(\omega/c)\,{\rm Im}\,\alpha$ is the scattering cross section. Physically, the light pressure force requires transfer of light momentum to the particle. Hence, light is either absorbed or scattered. Namely, for a small spherical particle with the radius R one has

$$\alpha = \frac{\alpha_0}{1 - \frac{2{\rm i}q^3}{3}\alpha_0}, \quad \alpha_0 = R^3\frac{\varepsilon - 1}{\varepsilon + 2}\,. \tag{22}$$

We see, that even for dielectric particle with ${\rm Im}\,\alpha = 0$ we still have ${\rm Im}\,\alpha \neq 0$, since energy losses are possible due to emission to the free space. However, for small enough particle the radiative corrections are weak, ${\rm Im}\,\alpha \ll {\rm Re}\,\alpha$, so the gradient force provides the dominant contribution and ensures the trapping.

Given that $\boldsymbol{B}^* = {\rm i}(c/\omega)\,{\rm rot}\,\boldsymbol{E}^*$ we can rewrite Eq. (20) as (Chaumet and Nieto-Vesperinas, 2000)

$$\boldsymbol{F} = 2\,{\rm Re}\,[\alpha(\boldsymbol{E}\cdot\nabla)\boldsymbol{E}^* + \alpha\boldsymbol{E}\times(\nabla\times\boldsymbol{E}^*)] \tag{23}$$

or

$$F_\mu = 2\,{\rm Re}\left[\alpha E_\nu \frac{\partial E_\nu^*}{\partial x_\mu}\right]\,,\ \text{where}\ \mu,\nu = x,y,z\,. \tag{24}$$

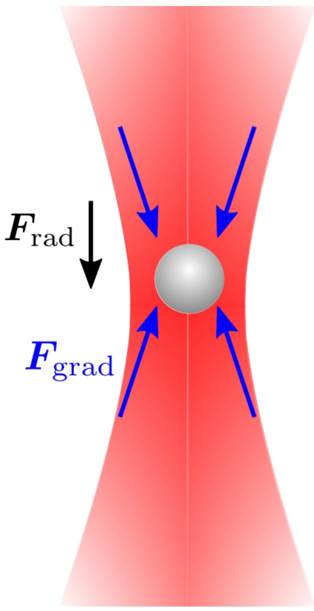

FIG. 4 Illustration of optical trapping of dielectric particle in the focus of laser beam. Directions of radiation pressure and gradient force are shown.

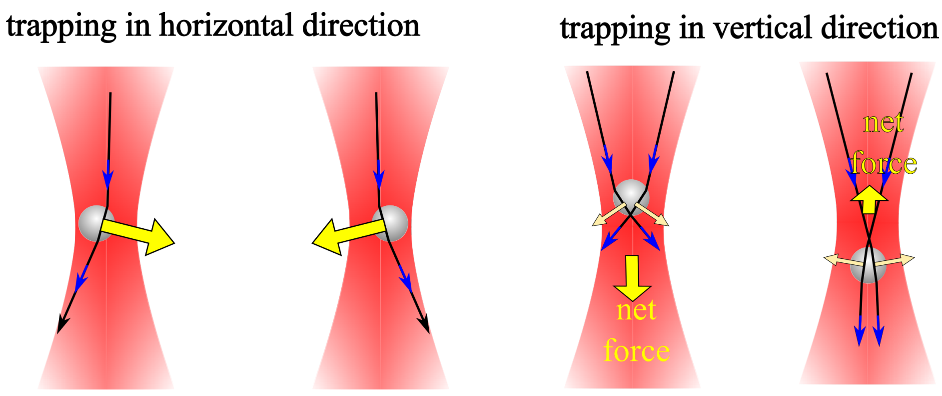

FIG. 5 Explanation of the optical trapping in the focus of the laser beam by ray tracing. Yellow arrow shows the direction of the net force.

B. Radiation trap in the ray optics language

Radiation trapping of particles, that are larger than the wavelength, could be understood using geometrical optics. Namely, it is possible to show that the particle is attracted to the focus of laser beam independent of its initial position by tracing the refraction of the rays, see Fig. 5.

C. Abraham-Minkowsky controversy for the surface of liquid and its resolution

Let us consider the controversy in Fig. 2 in more detail. This section loosely follows the discussion of Fig. 2 in Ref. (Gordon, 1973). In fact, there is no controversy, it arises only because the matter and electromagnetic field degrees of freedom are sometimes separated in different ways, see the review (Pfeifer *et al.*, 2007) for more details. One arrives to different answers only when neglecting some terms or forces. If all the terms are properly taken into account, the result will be correct independent of the approach. Let us try to calculate again the force acting on the surface of the liquid with the refractive index $n \approx 1$, $n-1 \ll 1$ by making such different calculations.

"Abraham approach" (neglecting the lateral gradient force)

We assume that the wave packet in the medium has total momentum density (mechanical+electromagnetic).

$$G_z^{(\mathrm{medium})} = \frac{nE_xH_y}{4\pi c} = \frac{(n-1)E_xH_y}{4\pi c} + \frac{E_xH_y}{4\pi c}\,, \tag{25}$$

where $M = (n-1)E_x H_y/(4\pi c)$ is the mechanical momentum density, see discussion of Eq. (18). Here, $\frac{E_x H_y}{4\pi c}$ is the Abraham electromagnetic momentum density, that is the same in vacuum and corresponds to the stress tensor (7) that also is the same as in vacuum. Now, if we consider the thin layer near the surface, the momentum that enters the layer from air per unit area and unit time is $\mathrm{d}p/\mathrm{d}t = cG_z^{(\mathrm{vac})} = E_x H_y/(4\pi)$. The momentum that leaves the layer and goes further inside the medium per unit time is $c/nG_z^{(\mathrm{medium})} = E_x H_y/(4\pi)$, i.e. it is the same ($E_x$ and H_y are continuous at the surface). Hence, no momentum is transferred to the layer at the surface and no force acts. This is consistent with the results of three other approaches: (i) Lorentz force Eq. (5) acting upon the infinitely thin layer is zero, (ii) the Abraham stress tensor Eq. (7) is continuous at the boundary and that (iii) no momentum is left in a given point of the gas after the pulse has passed, see Fig. 3.

The momentum per single photon i can be obtained by dividing the *total* momentum density by the photon flux. Since the photon flux is the same in medium and in vacuum, and the total momentum flux $c/nG_z^{(\mathrm{medium})}$ is also the same in medium and in vacuum, we conclude that the momentum per single photon in the medium and in vacuum is also the same and equal to $\hbar\omega/c$. This is again consistent: the photons have the same momenta to the left and right from the surface layer, their flux is conserved, so no momentum is transferred to the layer.

Lateral gradient force

The previous discussion ignores the ponderomotive force due to finite *width* of the beam. In fact,

$$\overline{\boldsymbol{f}(t)} = \nabla U, \quad U = \frac{1}{2}\alpha N\overline{E^2(t)} \approx \frac{n-1}{4\pi}\overline{E^2(t)} = (n-1)\frac{J}{c}, \tag{26}$$

where $J = c\overline{E^2(t)}/(4\pi)$ is the light intensity. Hence, there is an extra pressure U in the parts of liquid inside the beam that "pushes" the liquid to the surface in the vertical direction, see Fig. 6. The total vertical pressure acting at the surface is then equal to

$$f_z^{(\mathrm{total})} = -(n-1)J/c\,. \tag{27}$$

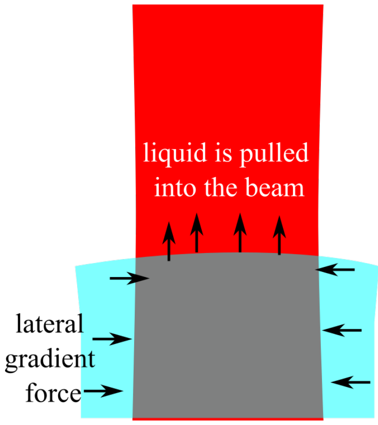

FIG. 6 Explanation of water being pulled into the laser beam of finite width due to the lateral gradient force

Lateral ponderomotive force

The same pressure Eq. (27) could also be obtained by calculating the flux of *Minkowsky momentum*. The density of Minkowsky momentum is given by

$$\boldsymbol{G}^{(\mathrm{Mink})}=\frac{\boldsymbol{D}\times\boldsymbol{B}}{4\pi c}\,. \tag{28}$$

and corresponds to the stress tensor (Zangwill, 2013),

$$\sigma_{\alpha\beta}^{(\mathrm{Mink})}=\frac{1}{4\pi}\left[\varepsilon E_\alpha E_\beta+\mu H_\alpha H_\beta-\frac{\delta_{\alpha\beta}}{2}(\varepsilon E^2+\mu H^2)\right] \tag{29}$$

where we assume isotropic medium with the permittivity ε and the permeability μ. The momentum, leaving the thin surface layer per unit time, following from Eq. (28) is $c/nG_z^{(\mathrm{Mink})}=nE_xH_y/(4\pi)$. The rate of momentum, entering the layer is still $E_xH_y/(4\pi)$, as in the previous section. Hence, the momentum transferred to the layer is $-(n-1)E_xH_y/(4\pi)=-(n-1)J/c$, in agreement with Eq. (27). This can be understood as a difference between the momentum of photon $\hbar\omega/c$, coming into the layer and the momentum $\hbar\omega n/c$ going from the layer to the medium.

To summarize the discussion, *naive Abraham approach yields zero, but lateral gradient force adds $-(n-1)J/c$ which is the same as the naive Minkowsky approach.* This coincidence is not accidental, see discussion in (Gordon, 1973) (Landau and Lifshitz, 1974), (Pfeifer *et al.*, 2007).

This section demonstrates, that macroscopic electrodynamics by itself should not be regarded as a fully self-consistent theory; Maxwell equations in the medium should be ac-

		Abraham	Minkowsky
Energy density		$\frac{\varepsilon E^2+H^2}{8\pi} \equiv \frac{\varepsilon E^2}{4\pi}$	
Momentum density	Electromagnetic	$\frac{\boldsymbol{E}\times\boldsymbol{H}}{4\pi c}$	undefined
	Mechanical	$(n-1)\frac{\boldsymbol{E}\times\boldsymbol{H}}{4\pi c}$	undefined
	Total	$\frac{n\boldsymbol{E}\times\boldsymbol{H}}{4\pi c} \equiv \frac{\varepsilon E^2}{4\pi c}\frac{\boldsymbol{k}}{k}$	$\frac{n^2\boldsymbol{E}\times\boldsymbol{H}}{4\pi c}$
Single photon energy		$\hbar\omega$	
Photon momentum		$\frac{\hbar\omega}{cn}$	$\frac{\hbar\omega}{c}n$
Photon + mechanical momentum		$\frac{\hbar\omega}{c}$	

TABLE I Comparison of Abraham and Minkowsky approaches for a plane wave propagating in a medium with refractive index n.

companied by material relations distinguishing a liquid from a solid, etc. More details on the Maxwell stress tensor in a medium are given in (Landau and Lifshitz, 1974).

III. MORE READING

A. Optical pressure

Resolution of Abraham-Minkowsky controversy and different confusions with Maxwell stress tensor: recent review and textbook: (Pfeifer *et al.*, 2007), (Zangwill, 2013),

Very pedagogical analysis (Gordon, 1973), pioneering experiment: (Ashkin and Dziedzic, 1973).

Very detailed analysis of different contributions for light pressure on a dielectric half-space (Loudon, 2002).

Literature in this field can be quite contradictory and should be read with exreme caution, even the reviews by renowned experts. For instance, a recent review (Toptygin and Levina, 2016) presents the stress tensor Eq. (49), that is called Minkowsky stress tensor, but is actually different from the Minkowsky stress tensor given in all other literature, e.g. (Pfeifer *et al.*, 2007; Zangwill, 2013).

Generalization of stress tensors and optical forces in case when the permittivity depends on frequency or wavevector (time and spatial dispersion) is still a subject of active studies, see e.g. (Bliokh *et al.*, 2017a,b).

B. Optical trapping

First experiment of Ashkin for dielectric particles: (Ashkin *et al.*, 1986)
General derivations: (Novotny and Hecht, 2012)

Force acting on a small dielectric and magneto-dielectric particle: (Chaumet and Nieto-Vesperinas, 2000; Chaumet and Rahmani, 2009)

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Lecture 2. Mechanisms of Optomechanical Interaction in Microcavities

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Contents

In the previous Lecture we considered how light does affect the motion of matter. Now we are going the consider the opposite effect: how does the motion of matter affects the photonic modes of the system. We will first provide a general introduction to the mechanisms of optomechanical interaction (Sec. I), briefly discuss semiconductor microcavities (Sec. I.D) and then discuss how to probe optomechanical interactions in optical experiments (Sec. II).

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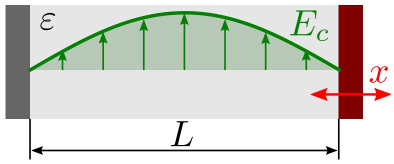

FIG. 1 Cavity with a moving mirror

I. INTERACTION MECHANISMS

For simplicity we consider a simple Fabry-Perot cavity filled with a medium with permittivity ε. Our goal is to examine the effect of the the deformation on the cavity resonant frequency.

A. Geometric mechanism

In this case the mode frequency shifts due to the movement of the mirror:

$$\omega_c = \frac{\pi c}{(L+x)\sqrt{\varepsilon}} \approx \frac{\pi c}{L\sqrt{\varepsilon}}(1-\frac{x}{L}) = \omega_c^{(0)} - g_{\mathrm{geom}}x, \quad g_{\mathrm{geom}} = \omega_c^{(0)}\frac{1}{L}\,. \tag{1}$$

The value of geometric coupling constant can be roughly estimated as 10^{15} Hz/1000 nm = 1 THz/nm.

B. Photoelastic mechanism

This mechanism is due to the modification of the permittivity of the cavity due to the deformation of the cavity material. We introduce the strain

$$u = \frac{x}{L}\,, \tag{2}$$

Hence, Eq. (1) has to be modified to

$$\omega_c = \omega_c^{(0)} - (g_{\mathrm{geom}} + g_{\mathrm{photoel}})x, \tag{3}$$

where

$$g_{\mathrm{photoel}} = \omega_c^{(0)}\frac{1}{2\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}u}\frac{1}{L}\,. \tag{4}$$

The ratio of photoelastic and geometric contributions can be estimated as

$$\frac{g_{\mathrm{photoel}}}{g_{\mathrm{geom}}} \sim \frac{1}{2\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}u}\,. \tag{5}$$

The photoelastic contribution dominates at the material resonances. For example, let us consider the cavity filled by a semiconductor with the excitonic resonance at the frequency ω_0. The strain leads to the shift of the semiconductor band gap, which is described by the *deformation potential* Ξ (for a typical semiconductor $\Xi \sim 10$ eV). The permittivity has a resonance at the frequency ω_0:

$$\varepsilon = \varepsilon_b \left(1 + \frac{\omega_{LT}}{\omega_0 + \Xi u - \omega - \mathrm{i}\Gamma} \right) , \tag{6}$$

ε_b is the background permittivity, ω_{LT} is the longitudinal-transverse splitting describing the light-exciton coupling strength, Γ is the resonance damping. Hence, the photoelastic term has a double resonance at the exciton frequency:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}u} \propto \frac{1}{(\omega_0 - \omega - \mathrm{i}\Gamma)^2} \, . \tag{7}$$

In experiment the value of $(1/\varepsilon)\mathrm{d}\varepsilon/\mathrm{d}u \sim 10^5$ has been observed at the excitonic resonances in quantum wells at low temperatures (Jusserand *et al.*, 2015) .

C. Dissipative coupling

Other interaction mechanisms are possible in principle. For instance, mechanical vibrations can modulate the damping rate κ of optical modes. This effect is termed as *dissipative coupling* (Li *et al.*, 2009; Kyriienko *et al.*, 2014). Values for the waveguide system:(Li *et al.*, 2009)

$$\gamma_{\mathrm{om}} = \frac{\mathrm{d}\kappa}{\mathrm{d}x} = -27 \text{ MHz/nm}. \tag{8}$$

D. Semiconductor microcavities

A nice platform to study the optomechanical effect on the nanoscale is a micropillar, with of two Bragg mirrors, see Fig. 2. Each Bragg mirror contains a stack of alternating GaAs and AlGaAs layers. The key observation of Ref. (Fainstein *et al.*, 2013) was that the ratio of refractive indices of GaAs and AlAs is approximately equal to the ratio of sound velocities, $c_2/c_1 = s_2/s_1 \approx 1.2$. Hence, the same structure acts like a Bragg mirror both for light and sound.The advantages of such structure are

- Ultra-High frequencies of confined acoustic phonon modes: $\Omega/2\pi$ up to 100 GHz (Anguiano *et al.*, 2017) (estimation: $s/d \sim 10^6$ cm/s/100 nm = 100 GHz).

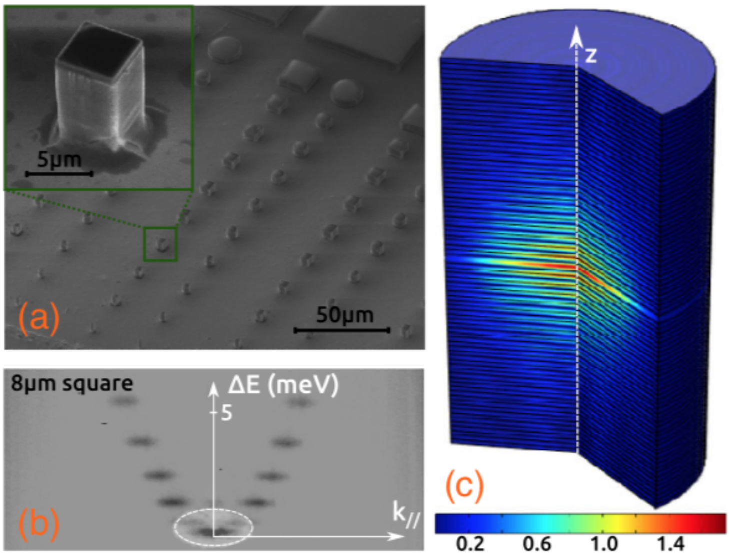

FIG. 2 From Ref. (Anguiano *et al.*, 2017). (a) SEM images of an array of circular and square pillars with lateral sizes ranging from 50 to 1 μm. The inset presents a zoom on a 5 μm square pillar. (b) k-space image of the optical cavity modes for a square pillars of 8 μm lateral size. The shaded ellipse represents the profile (energy broadening and angular dispersion) of the pump and probe laser pulses. (c) Spatial distribution of volumetric strain associated with a confined mechanical mode around 19 GHz, calculated using finite element methods.

- Strong photoelastic coupling: $g_{\rm photoel} \sim 83$ THz/nm, $g_{\rm geom} \sim 4$ THz/nm (Fainstein *et al.*, 2013) (10^3 times larger than for integrated waveguides (Li *et al.*, 2009))
- planar scalable integrated technology at nanoscale .

Very important recent work: "Polariton-driven phonon laser", (Chafatinos *et al.*, 2020)

II. PROBING OPTOMECHANICAL INTERACTION FOR WEAK COUPLING WITH PHONONS

1. Time-dependent transmission

How can one probe the optomechanical interaction in experiment? The first approach is the pump-probe reflection experiment. In this case the first pump pulse excites the deformations of the structure. For instance, it can heat the sample, which leads to the expansion of the lattice and coherent excitation of different mechanical modes. Let us consider how the mechanical vibrations modify the reflection coefficient. Suppose, that we work near an optical resonance (e.g. a Fabry-Perot one), so the amplitude transmission coefficient of light through the cavity reads

$$t(\omega)=\frac{\mathrm{i}\Gamma_0}{\omega_c-\omega-\mathrm{i}(\Gamma+\Gamma_0)}\,. \tag{9}$$

Here ω_c is the resonance frequency, and the parameters Γ_0 and Γ characterize the resonance strength and the width. Due to the mechanical vibrations with the amplitude x_0 and the frequency Ω the cavity frequency oscillates in time as

$$\omega_c=\omega_c^{(0)}+g(x_0\mathrm{e}^{-\mathrm{i}\Omega t}+x_0^*\mathrm{e}^{\mathrm{i}\Omega t})\,. \tag{10}$$

We assume that the structure is illuminated by a monochromatic pulse ,

$$E_0(t)=E_0\mathrm{e}^{-\mathrm{i}\omega t}=\int\frac{\mathrm{d}\omega}{2\pi}E_0(\omega)\mathrm{e}^{-\mathrm{i}\omega t},\quad E_0(\omega)=2\pi E_0\delta(\omega-\omega_0)\,. \tag{11}$$

The transmitted field can be then presented as

$$\begin{aligned}
E_t(t)&=\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}t(\omega)E_0(\omega)=\\
&=\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}t^{(0)}(\omega)E_0(\omega)+\int\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}t(\omega)}{\mathrm{d}\omega_c}g(x_0\mathrm{e}^{-\mathrm{i}\Omega t}+x_0^*\mathrm{e}^{\mathrm{i}\Omega t})\mathrm{e}^{-\mathrm{i}\omega t}E_0(\omega)=\mathrm{e}^{-\mathrm{i}\omega_0 t}[E_t^{(0)}+E_t^{(1)}(t)]\,,
\end{aligned} \tag{12}$$

where

$$E_t^{(0)}(t)=t^{(0)}(\omega)E_0,\quad E_t^{(1)}(t)=\frac{\mathrm{d}t}{\mathrm{d}\omega_0}E_0g(x_0\mathrm{e}^{-\mathrm{i}\Omega t}+x_0^*\mathrm{e}^{+\mathrm{i}\Omega t})\,. \tag{13}$$

Hence, the transmitted wave will not be monochromatic. Due to the modulation of the cavity parameters it will contain the response at the anti-Stokes and Stokes frequencies .

$$\omega_{\mathrm{aS}}=\omega_0+\Omega,\quad \omega_{\mathrm{S}}=\omega_0-\Omega\,. \tag{14}$$

A. Scattering on phonons

Another approach is to study light scattering on thermal vibrations instead of reflection from coherent vibrations (Jusserand *et al.*, 2015) . In this case average amplitude of vibrations in Eq. (13) is zero,

$$\langle x_0\rangle=0 \tag{15}$$

however the intensity of vibrations is nonzero:

$$\langle |x_0|^2 \rangle \propto \begin{cases} N_\Omega + 1 & \text{(Stokes)} \\ N_\Omega & \text{(anti-Stokes)} \end{cases} \tag{16}$$

In practice the phonon frequency is much smaller than temperature: $T = 300$ K corresponds to 25 meV = 6 THz, while the largest photon frequency are about 0.1 THz. Hence, spontaneous photon emission can be neglected, $T \gg \hbar\Omega$ and

$$N_\Omega = \frac{1}{\mathrm{e}^{\hbar\Omega/T} - 1} \approx \frac{T}{\hbar\Omega} \gg 1 \text{ if } T \gg \hbar\Omega\,. \tag{17}$$

Let us develop a more detailed theory of light scattering in a microcavity. We consider the Hamiltonian

$$H = \omega_c c^\dagger c + \Omega_m a^\dagger a - g_c c^\dagger c (a + a^\dagger)\,. \tag{18}$$

where c is the photon creation operator, a is the creation operator of the phonons with the frequency Ω, and g_c is the optomechanical coupling constant. Next, we write the Heisenberg equations of motion $\dot{c} = \mathrm{i}[H, c]$, $\dot{a} = \mathrm{i}[H, a]$ for the operators c and a:

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\mathrm{i}[\omega_c - g_c(a + a^\dagger)]c \tag{19}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mathrm{i}\Omega_m a - \mathrm{i}g_c c^\dagger c a\,. \tag{20}$$

In this section we will solve these equations in the classical regime, when the photon and photon occupations numbers are large. In this case the operators can be replaced by complex numbers.

We will also include the damping for the cavity mode Γ_c and will first assume that the phonon mode oscillates monochromatically as

$$a(t) = u\mathrm{e}^{-\mathrm{i}\Omega_m t}\,. \tag{21}$$

The first of Eqs. (19) then transforms to

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\mathrm{i}[\omega_c + g_c(u\mathrm{e}^{-\mathrm{i}\Omega_m t} + u^* \mathrm{e}^{\mathrm{i}\Omega_m t})]c - \Gamma_c c - \mathrm{i}E_0 \mathrm{e}^{-\mathrm{i}\omega t}\,, \tag{22}$$

where E_0 is the amplitude of the incoming field with the frequency ω. Our goal is to solve Eq. (22) for small amplitude g_c, when the interaction can be considered as a perturbation. We will also transform Eq. (22) in the Fourier domain. Then the cavity field will oscillate at the initial frequency ω and also at Stokes and anti-Stokes frequencies

$$\omega_{\mathrm{aS}} = \omega + \Omega, \quad \omega_{\mathrm{S}} = \omega - \Omega\,. \tag{23}$$

as

$$c(t) = c_0 \mathrm{e}^{-\mathrm{i}\omega t} + c_{\mathrm{aS}} \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{aS}} t} + c_{\mathrm{S}} \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{S}} t}\,, \tag{24}$$

where the field c_0 does not depend on the phonon amplitude and $c_{\mathrm{aS}}, c_{\mathrm{S}} \propto g_c$.

$$\begin{aligned} (\omega_c - \omega - \mathrm{i}\Gamma_c) c_0 &= -E_0 \\ (\omega_c - \omega_{\mathrm{aS}} - \mathrm{i}\Gamma_c) c_{\mathrm{aS}} &= -g_c u c_0 \\ (\omega_c - \omega_{\mathrm{S}} - \mathrm{i}\Gamma_c) c_{\mathrm{S}} &= -g_c u^* c_0\,. \end{aligned} \tag{25}$$

Solving Eqs. (25) we find

$$c_{\mathrm{aS}} = \frac{g E_0 u}{(\omega_c - \omega_{\mathrm{aS}} - \mathrm{i}\Gamma_c)(\omega_c - \omega - \mathrm{i}\Gamma_c)}, \quad c_{\mathrm{S}} = \frac{g E_0 u^*}{(\omega_c - \omega_{\mathrm{S}} - \mathrm{i}\Gamma_c)(\omega_c - \omega - \mathrm{i}\Gamma_c)}\,. \tag{26}$$

The spectrum of the scattered field can be presented by taking the square of the amplitudes c_{aS}, c_{S}:

$$I = g^2 |E_0|^2 \frac{|u|^2}{[(\omega_c - \omega')^2 + \Gamma_c^2][(\omega_c - \omega)^2 + \Gamma_c^2]} \tag{27}$$

where ω' is the scattered light frequency equal to ω_{aS} or ω_{S}. In the stochastic case we can just replace $|u|^2$ by its thermal average $\langle a^\dagger a \rangle \propto T/\omega$, see Eq. (16). We see from Eq. (27) that the scattered light has two resonances, for incident and scattered photons ω and ω', respectively, see also Fig. 3.

In this consideration we have considered vibrations classically. This consideration is valid when the temperature T is much larger than the characteristic phonon frequency Ω, $k_B T \gg \hbar\Omega$. We note, that temperature of 300K corresponds to the frequency of $\approx 6THz$, which is much larger than the typical vibration frequency. Hence, at room temperature the classical approximation is almost always valid, but it can break at lower temperatures. In this case we should slightly adjust Eq. (21) and replace c-numbers by the operators:

$$\hat{a}(t) = \hat{u} \mathrm{e}^{-\mathrm{i}\Omega_m t}, \quad \hat{x}(t) = \hat{a}(t) + \hat{a}^\dagger(t)\,. \tag{28}$$

Here $\hat{u}^\dagger$ and $\hat{u}$ are proportional to the phonon creation and destruction operators, so

$$\langle \hat{u}^\dagger \hat{u} \rangle = N_\Omega, \quad \langle \hat{u} \hat{u}^\dagger \rangle = N_\Omega + 1, \quad N_\Omega = \frac{1}{\mathrm{e}^{\hbar\Omega/T} - 1}\,, \tag{29}$$

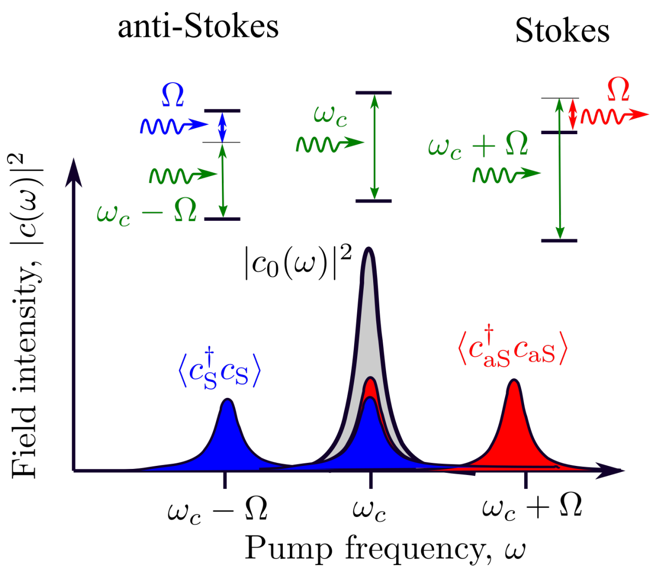

FIG. 3 Schematic illustration of the Stokes and anti-Stokes light scattering spectra in a cavity, Eq. (27).

where the angular brackets denote the averaging over the equilibrium distribution function of the phonons. In case when $T \gg \hbar\Omega$ we get $N_\Omega \gg 1$ and the difference between N_Ω and $N_\Omega + 1$ (spontaneous emission) can be neglected. Our whole derivation remains valid also in the quantum case. The only difference is that in Eq. (26) we should replace u in $c_{\rm aS}$ by $\hat{u}$, and u^* in $c_{\rm S}$ by $u^\dagger$. As a result, the intensities of anti-Stokes and Stokes light will be proportional to $\langle \hat{u}^\dagger \hat{u}\rangle = N_\Omega$ and $\langle \hat{u}\hat{u}^\dagger\rangle = N_\Omega + 1$, respectively. At zero temperature, when the phonon number N_Ω is zero, we have only spontaneous phonon emission (Stokes processes) and do not have phonon absorption (anti-Stokes processes).

Resonant optomechanical coupling in microcavities

We will now assume that the microcavity has an embedded quantum well with an excitonic resonance. Hence, the Hamiltonian Eq. (18) has to be extended to include photon-exciton coupling and phonon-exciton coupling. The resulting Hamiltonian describing interaction of exciton (b) with cavity photon (c) and phonon (a) modes reads

$$H = \omega_x b^\dagger b + \omega_c c^\dagger c + \Omega_m a^\dagger a + \frac{\omega_R}{2}(c^\dagger b + b^\dagger c) + g_x b^\dagger b(a + a^\dagger) + g_c c^\dagger c(a + a^\dagger)\,. \tag{30}$$

Here ω_R is Rabi frequency characterizing the strength of exciton-photon interaction; the optomechanical constant for bare exciton g_x is given by

$$g_x = \Xi u_0$$

with $\Xi \approx 9\,\text{eV}$ being the deformation potential constant and $u_0 = k x_{\text{ZPF}} \approx 10^{-8}$ the zero-point deformation (Jusserand *et al.*, 2015).

Instead of Eq. (24) we write

$$c(t) = c_0 \mathrm{e}^{-\mathrm{i}\omega t} + c_{\text{aS}} \mathrm{e}^{-\mathrm{i}\omega_{\text{aS}} t} + c_{\text{S}} \mathrm{e}^{-\mathrm{i}\omega_{\text{S}} t} \tag{31}$$

$$b(t) = b_0 \mathrm{e}^{-\mathrm{i}\omega t} + b_{\text{aS}} \mathrm{e}^{-\mathrm{i}\omega_{\text{aS}} t} + b_{\text{S}} \mathrm{e}^{-\mathrm{i}\omega_{\text{S}} t}\,, \tag{32}$$

and the generalization of Eq. (33) reads

$$\begin{aligned}
(\omega_c - \omega - \mathrm{i}\Gamma_c) c_0 + g b_0 &= E_0\,, \\
(\omega_X - \omega - \mathrm{i}\Gamma_X) b_0 + g c_0 &= 0\,, \\
(\omega_c - \omega_{\text{aS}} - \mathrm{i}\Gamma_c) c_{\text{aS}} + g b_{\text{aS}} &= -g_c u c_0\,, \\
(\omega_X - \omega_{\text{aS}} - \mathrm{i}\Gamma_X) b_{\text{aS}} + g c_{\text{aS}} &= -g_x u b_0\,.
\end{aligned} \tag{33}$$

and similarly for Stokes scattering. Solving Eq. (33) we find the the intensity of Raman scattering

$$I \propto \left| \frac{g_x \omega_R^2/4 + g_c(\omega - \omega_x + \mathrm{i}\Gamma_x)(\omega' - \omega_x + \mathrm{i}\Gamma_x)}{[(\omega - \omega_x + \mathrm{i}\Gamma_x)(\omega - \omega_c + \mathrm{i}\Gamma_c) - \omega_R^2/4][(\omega' - \omega_x + \mathrm{i}\Gamma_x)(\omega' - \omega_c + \mathrm{i}\Gamma_c) - \omega_R^2/4]} \right|^2 . \tag{34}$$

where the incoming and outgoing light frequencies are related by $|\omega - \omega'| = \Omega_m$.

One can rewrite Eq. (34) as

$$I \propto \left| \frac{g_{\text{eff}}}{[\omega - \omega_c + \mathrm{i}\Gamma_c - \frac{\omega_R^2/4}{\omega - \omega_x + \mathrm{i}\Gamma_x}][\omega' - \omega_c + \mathrm{i}\Gamma_c - \frac{\omega_R^2/4}{\omega' - \omega_x + \mathrm{i}\Gamma_x}]} \right|^2 , \tag{35}$$

where the effective optomechanical constant reads

$$g_{\text{eff}} = g_c + \frac{g_x \omega_R^2/4}{(\omega - \omega_x + \mathrm{i}\Gamma_x)(\omega' - \omega_x + \mathrm{i}\Gamma_x)}\,.\,, \tag{36}$$

In the *weak coupling regime*, the resonant contributions in denominator of Eq. (35) are small and can be neglected. Then the is given by

$$I \propto \left| \frac{g_{\text{eff}}}{(\omega - \omega_c + \mathrm{i}\Gamma_c)(\omega' - \omega_c + \mathrm{i}\Gamma_c)} \right|^2 . \tag{37}$$

Comparing Eq. (37) with Eq. (27) we see that the optomechanical interaction constant is replaced from g_c to the function (36) that has an additional resonant term.

In the *strong coupling regime*, the resonance of Eq. (36) can not be exploited to increase the observable quantities such as scattering intensity. The reasons are as follows. (a) At the bare exciton frequency ω_x the large value of $g^{\rm eff}$ is compensated by the resonant terms in the denominator of Eq. (35). (b) At the polariton resonance frequency ω_p the detuning from the bare exciton resonance is large ($\sim \omega_R \gg \Gamma_x$), so the relevant value of $g^{\rm eff}$ is relatively small. In Ref. (Jusserand *et al.*, 2015) we used Eq. (36) but have forgotten about the resonant terms in the denominator of Eq. (35), leading to the detuning of the polariton resonances from bare exciton one.

In particular, for $\Gamma_x, \Gamma_c \ll \omega_R$, the maximum intensity is achieved when both ω and ω' are in the vicinity of upper or lower polariton resonances $\omega_p = [\omega_x - \omega_c \pm \sqrt{(\omega_x - \omega_c)^2 + \omega_R^2}]/2$, see Ref. (Kavokin *et al.*, 2006) for more details on polaritons in microcavities.

Then Eq. (34) simplifies and assumes the form [cf. Eq. (2) in Rozas *et al.*, PRB **90**, 201302 (2014)]

$$I \propto \left| C_c \frac{g_{\rm pol}}{(\omega - \omega_p + {\rm i}\Gamma_p)(\omega' - \omega_p + {\rm i}\Gamma_p)} \right|^2 , \tag{38}$$

where the Hopfield coefficients are given by

$$C_x = \frac{\omega_R^2/4}{(\omega_p - \omega_x)^2 + \omega_R^2/4} , \tag{39}$$

$$C_c = \frac{\omega_R^2/4}{(\omega_p - \omega_c)^2 + \omega_R^2/4} , \tag{40}$$

and the effective optomechanical coupling constant for polariton is given by the weighted average of optomechanical coupling constants for excitons and photons,

$$g_{\rm pol} = C_x g_x + C_c g_c \, . \tag{41}$$

Physically, the Hopfield coefficients with $C_x + C_c = 1$ tell us which about the photonic and excitonic contributions into the polariton modes. Since g_x is proportional to the larger deformation potential of exciton $\Xi \sim 10$ eV, there is still a room to improve the scattering by using resonant optomechanical coupling.

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Lecture 3. Optomechanical Backaction

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(Dated: April 21, 2020)

Contents

I. OPTOMECHANICAL SPRING EFFECT

In the previous Lecture we have analyzed the effect of the mirror vibrations of light. Namely, the vibrations lead to appearance of Stokes and anti-Stokes light scattering in a cavity. In this Lecture we go further and take into account the *optomechanical backaction*: effect of light on the movement of the mirror.

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\mathrm{i}(\omega_c - gx)c - \Gamma_c c + E_0 \mathrm{e}^{-\mathrm{i}\omega t} \tag{1}$$

(note the different sign of g)

$$|c|^2 = \frac{E_0^2}{\Gamma_c^2 + (\omega + gx - \omega_c)^2} \tag{2}$$

$$F_{\text{rad}}(x) = g|c|^2(x) = F_{\text{rad}}(x=0) - kx, \quad k \propto (\omega - \omega_c) \tag{3}$$

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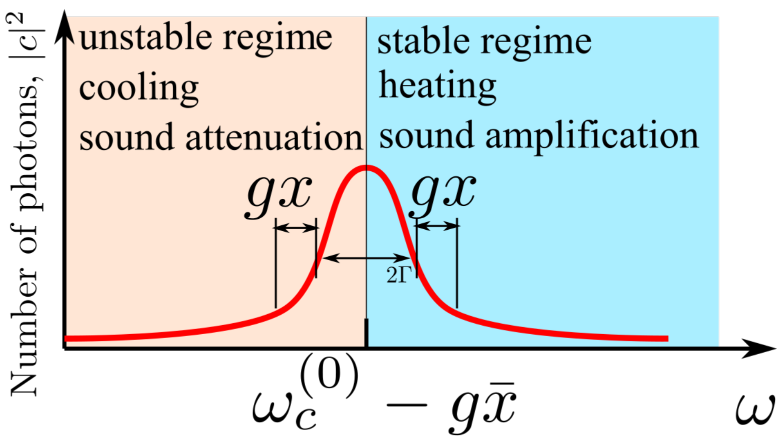

FIG. 1 Optical force and number of photons in the cavity: classical picture (adapted from Ref. (Aspelmeyer *et al.*, 2014))

For negative detuning, $\omega < \omega_c$, we get $k < 0$ which can lead to instability, see Fig. 1. Namely, at negative detuning increase of x due to radiation pressure leads to decrease of ω_c. Hence, detuning decreases, hence, radiative pressure increases and ω_c decreases stronger. This means instability. For positive detuning the decrease of ω_c leads to increase of detuning and smaller radiation pressure, hence, the system remains stable.

This argument can be elaborated further:

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -m\Omega^2 x + g|c|^2 = -\frac{\mathrm{d}U}{\mathrm{d}x} \tag{4}$$

$$U(x) = \frac{m\Omega^2 x^2}{2} - \frac{E_0^2}{\Gamma_c}\arctan\left(\frac{\omega - \omega_c + gx}{\Gamma_c}\right) \tag{5}$$

II. OPTOMECHANICAL COOLING AND HEATING

A. Heating and cooling: classical picture

Let us assume that the cavity vibrates at the frequency Ω and calculate, how the radiation pressure force affects these vibrations. Namely, our goal is to find the total work of the force per one cycle of vibrations ,

$$\oint F_{\mathrm{rad}}\mathrm{d}x = \int_0^{2\pi/\Omega} F_{\mathrm{rad}}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t, \quad x = x_0\cos\Omega t\,. \tag{6}$$

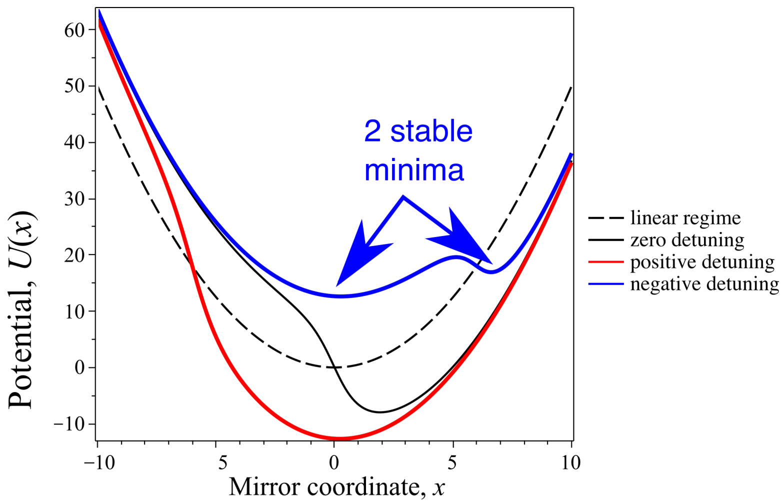

FIG. 2 Optomechanical potential Eq. (5). Black dashed curve: $E_0 = 0$, black solid curve: $E_0 = 3$,$\omega = \omega_c$, red curve: $\omega = \omega_c + 6g$, blue curve: $\omega = \omega_c - 6g$. Other calculation parameters are $m = g = \Gamma_c = \Omega = 1$.

At the first glance this work is zero ,

$$\int_0^{2\pi/\Omega} F_{\rm rad}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = g\Omega x_0 \int_0^{2\pi/\Omega} \cos\Omega t \sin\Omega t = 0\,. \tag{7}$$

However, here we have assumed that the cavity immediately reacts to the shift of the mirrors. In fact, there is some retardation due to the finite lifetime of the photons $\tau = 1/(2\Gamma_c)$. Let describe it in the simplest possible form: $F_{\rm rad}(t) \Rightarrow F_{\rm rad}(t-\tau)$. Then we obtain for the work per one cycle of vibrations

$$\oint F_{\rm rad}\frac{\mathrm{d}x}{\mathrm{d}t} = g\Omega x_0 \int_0^{2\pi/\Omega} \cos\Omega(t-\tau)\sin\Omega t \approx k\Omega x_0 \int_0^{2\pi/\Omega} (\cos\Omega t + \tau\Omega\sin\Omega t)\sin\Omega t = k\pi\tau\Omega x_0. \tag{8}$$

Comparing with Eq. (3) we see that for $\omega > \omega_c$ one has $k > 0$. Hence, the radiation performs positive work, and the cavity is heated due to the radiative pressure. In the opposite case $\omega < \omega_c$, radiation performs negative work, $\Rightarrow$ radiation cools the cavity (mechanical vibrations transfer energy to radiation).

More on radiative cooling in another context: (Chen *et al.*, 2016)

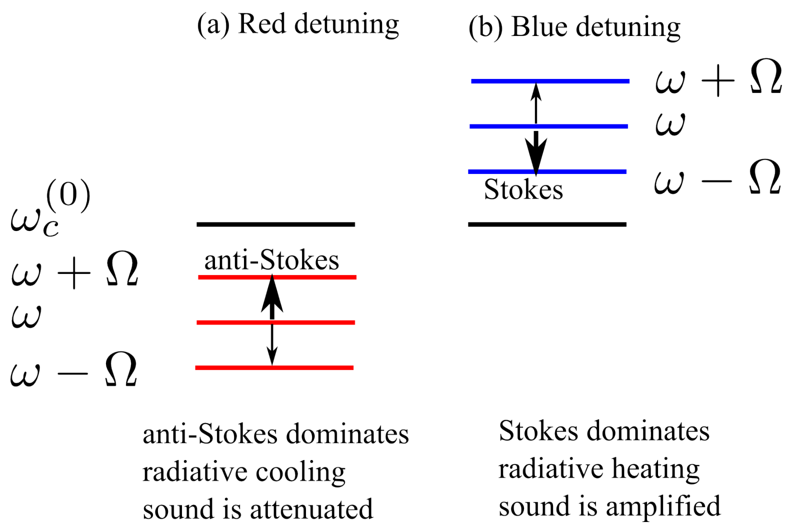

FIG. 3 Radiative heating and cooling

B. Scattering picture

The same results can be understood using the quantum-mechanical scattering picture, see Fig. 3. The pump incident upon a cavity can undergo Stokes and ant-Stokes scattering. The efficiency of this process is determined by the distance between the final state and the cavity resonance: the smaller is the distance the more efficient is the process. For $\omega < \omega_c$ anti-Stokes scattering with phonon absorption dominates $\Rightarrow$ radiative cooling. For $\omega > \omega_c$ the Stokes scattering is more efficient $\Rightarrow$ heating. The quantum-mechanical picture applies when the cavity mode linewidth is much less than the phonon mode energy.

III. MORE RIGOROUS DESCRIPTION OF COOLING AND HEATING

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\mathrm{i}[\omega_c - gx(t)]c - \Gamma_c c + E_0 \mathrm{e}^{-\mathrm{i}\omega t} \tag{9}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\Omega_m^2 x - 2\gamma_m \dot{x} + g|c|^2 \tag{10}$$

We have also introduced the damping Γ_c and Γ_m to the cavity and mechanical modes, respectively. Our goal is to find the eigenfrequency of mirror vibrations Ω. Let us introduce the vibration amplitude a and assume that $|\Omega - \Omega_m| \ll \Omega$.

$$x(t) = \bar{x} + a\mathrm{e}^{-\mathrm{i}\Omega t} + c.c., \quad \dot{x}(t) = -\mathrm{i}\Omega_m a\mathrm{e}^{-\mathrm{i}\Omega_m t} + c.c., \tag{11}$$

$$\ddot{x}(t)+\Omega_m^2 x=(\Omega_m^2-\Omega^2)a\mathrm{e}^{-\mathrm{i}\Omega t}+c.c\approx 2\Omega_m(\Omega_m-\Omega)a\mathrm{e}^{-\mathrm{i}\Omega t}+c.c \tag{12}$$

We stress, that in contrast to previous Lecture, we explicitly take into account the mirror dynamics instead of postulating it as $a(t)=\mathrm{e}^{-\mathrm{i}\Omega_m t}$. Now we assume, that the optomechanical interaction is weak and *linearize* Eqs. (9) with respect to this interaction:

$$c(t)=c_0(t)+\delta c(t)\,. \tag{13}$$

Namely, we assume that the strong amplitude c_0 is determined by the pump and does not depend on the vibrations, while the correction $\delta c(t)$ is due to vibrations. Substituting Eq. (13) into Eq. (9) we find the linearized equations:

$$\frac{\mathrm{d}c_0}{\mathrm{d}t}=-\mathrm{i}(\omega_c-g\bar{x})c_0-\Gamma_c c_0+E_0\mathrm{e}^{-\mathrm{i}\omega t} \tag{14}$$
$$\frac{\mathrm{d}\delta c}{\mathrm{d}t}=-\mathrm{i}(\omega_c-g\bar{x})c+\mathrm{i}gc_0(t)(a\mathrm{e}^{-\mathrm{i}\Omega t}+a^*\mathrm{e}^{\mathrm{i}\Omega t})-\Gamma_c\delta c \tag{15}$$
$$(\Omega_m-\Omega-\mathrm{i}\gamma_m)a=\frac{g}{2\Omega_m}(c_0^*\delta c+c_0\delta c^*)\mathrm{e}^{\mathrm{i}\Omega t} \tag{16}$$

Since the system of equations Eq. (14) is linear, we can try to look for the monochromatic solutions:

$$c(t)=c_0\mathrm{e}^{-\mathrm{i}\omega t}+c_{\mathrm{S}}\mathrm{e}^{-\mathrm{i}\omega_S t}+c_{\mathrm{aS}}\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{aS}}t},\quad \omega_S=-\Omega+\omega,\omega_{\mathrm{aS}}=\Omega+\omega\,. \tag{17}$$

Here Ω is the frequency of oscillations, and similarly to the previous Lecture we get Stokes and anti-Stokes modes. Substituting Eq. (17) into Eq. (14) we find

$$(\omega_c-g\bar{x}-\omega-\mathrm{i}\Gamma_c)c_0=-\mathrm{i}E_0\,, \tag{18}$$
$$(\omega_c-g\bar{x}-\omega_{\mathrm{S}}-\mathrm{i}\Gamma_c)c_{\mathrm{S}}=gc_0a^*\,, \tag{19}$$
$$(\omega_c-g\bar{x}-\omega_{\mathrm{aS}}-\mathrm{i}\Gamma_c)c_{\mathrm{aS}}=gc_0a\,, \tag{20}$$
$$(\Omega_m-\Omega-\mathrm{i}\gamma_m)a=\frac{g}{2\Omega}(c_{\mathrm{S}}^*c_0+c_{\mathrm{aS}}c_0^*)\,, \tag{21}$$

Linearized equations Eq. (18)–Eq. (20) present the central result of this Lecture. In the following we will consider different aspects of these equations. Expressing c_{S} and c_{aS} from Eqs. (19),(20) and substituting back into Eq. (21) we find

$$(\Omega_m+\Sigma-\Omega-\mathrm{i}\gamma_m)a=0. \tag{22}$$

Thus, the eigenfrequency is given by

$$\Omega=\Omega_m-\mathrm{i}\gamma_m+\Sigma \tag{23}$$

where

$$\Sigma = -\frac{g^2|c_0|^2}{2\Omega}\left(\frac{1}{\omega_c - g\bar{x} - \omega_{\rm S} + {\rm i}\Gamma_c} + \frac{1}{\omega_c - g\bar{x} - \omega_{\rm aS} - {\rm i}\Gamma_c}\right) = \delta\Omega_m - {\rm i}\delta\gamma_m, \tag{24}$$

is the optomechanical correction, describing modification of mechanical resonant frequency Ω_m and mechanical damping γ_m due to interaction with light:

$$\delta\Omega_m = \frac{g^2|c_0|^2}{2\Omega}\left(\frac{\omega_{\rm S} - \omega_c + g\bar{x}}{(\omega_c - g\bar{x} - \omega_{\rm S})^2 + \Gamma_c^2} + \frac{\omega_{\rm aS} - \omega_c + g\bar{x}}{(\omega_c - g\bar{x} - \omega_{\rm aS})^2 + \Gamma_c^2}\right) \tag{25}$$

$$\delta\gamma_m = \frac{g^2|c_0|^2\Gamma_c}{2\Omega}\left(-\frac{1}{(\omega_c - g\bar{x} - \omega_{\rm S})^2 + \Gamma_c^2} + \frac{1}{(\omega_c - g\bar{x} - \omega_{\rm aS})^2 + \Gamma_c^2}\right). \tag{26}$$

The optical spring correction becomes negative for $\omega \ll \omega_c$ which means instability. We also see that the correction to the damping of the phonons is proportional to the difference between Stokes and anti-Stokes scattering. If we pump in resonance with the Stokes mode, the mechanical damping decreases, because extra phonons are generated due to the scattering. This can lead to the "phonon lasing". At the anti-Stokes resonance the damping decreases.

In case of very bad cavity, when $\Gamma_c \gg \Omega$ we find

$$\delta\Omega_m = \frac{g^2|E_0|^2(\omega - \omega_c + g\bar{x})}{\Omega\Gamma_c^4} \tag{27}$$

which agrees with the result of expansion of Eq. (5) up to quadratic terms in x.

Numerical solutions of Eqs. (9) in the regime of optomechanical heating and cooling can be found on YouTube here: heating and cooling.

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Lecture 4. Optomechanical effects in continuous medium. Mandelstam-Brillouin scattering. Phonoritons

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(Dated: April 23, 2021)

Contents

In the previous lecture we have considered interaction of localized photon mode with localized vibrations. The goal of this lecture is to extend these concepts to the continuous medium and consider optomechanical effects in light-sound interaction. We will also introduce the concept of phonoritons (Ivanov and Keldysh, 1982): polaritons hybridized with phonons in the presence of external optical pump.

I. DERIVATION OF EQUATIONS FOR INTERACTION OF LIGHT WITH VIBRATIONS

The derivation below loosely follows A.V. Poshakinskiy and A.N. Poddubny, Phys. Rev. X **9**, 011008 (2019). We start with the Lagrangian describing interaction of light with sound in the form

$$\mathcal{L} = \mathcal{L}_{\rm em} + \mathcal{L}_u + \mathcal{L}_{int} + \mathcal{L}_j(j)\ , \tag{1}$$

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or, explicitly,

$$\mathcal{L} = \frac{1}{8\pi}\left(\frac{1}{c^2}(\partial_t A)^2 - (\partial_z A)^2\right) + \frac{\rho}{2}[(\partial_t u)^2 - s^2(\partial_z u)^2] + \frac{1}{c}j(z)A(z+u) + \mathcal{L}_j \,. \tag{2}$$

We assume that the electromagnetic field is described by the vector potential $\boldsymbol{A}(z,t) \parallel x$. The medium can vibrate longitudinally, $\boldsymbol{u}(z) \parallel z$. The key concept here is that the movement of the medium does not affect the material relationship, it stays the same in the moving frame. Thus, the current $j(z)$ just senses the potential in the shifted point $z+u$. For more details see the derivation in `[asymscat.pdf]`.

The Lagrangian $\mathcal{L}_j$ depends only on the current and enforces the local material relationship

$$j(z) \equiv j[A(z+u)] = \frac{\partial P(z+u)}{\partial t} = \chi\frac{\partial E(z+u)}{\partial t} = -\frac{\chi}{c}\frac{\partial^2 A(z+u)}{\partial t^2} \tag{3}$$

where χ is the medium susceptibility. Here we intend to neglect the derivative of u over t because these effects are relativistically small. We use the vector potential because A_x is Lorentz invariant in the reference frame moving along z.

A. Equation for light

In order to derive the equation for vector potential we write

$$A(z+u) \approx A(z) + u\partial_z A, \quad \mathcal{L}_{int} \approx \frac{1}{c}j[A(z) + u\partial_z A] \,. \tag{4}$$

The Langrange equation

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \frac{\partial A}{\partial t}} + \frac{\partial}{\partial x}\frac{\partial L}{\partial \frac{\partial A}{\partial z}} = \frac{\partial L}{\partial A} \tag{5}$$

then becomes

$$\frac{1}{4\pi}\left(\frac{1}{c^2}\partial_{tt}A - \partial_{zz}A\right) + \frac{1}{c}\partial_z(ju) = \frac{1}{c}j[A(z+u)] \tag{6}$$

or

$$\frac{1}{c^2}\partial_{tt}A - \partial_{zz}A = \frac{4\pi}{c}(1-\partial_z u)j(z) \,. \tag{7}$$

Physically, it means that due to increase of the volume of the crystal in $1+\partial_z u$ times the concentration of atoms decreases in $1-\partial_z u$ times and the polarizability decreases. Given Eq. (3) we can rewrite the equation for the vector potential as

$$\frac{\varepsilon(z,u)}{c^2}\frac{\partial^2 A}{\partial t^2} = \frac{\partial^2 A}{\partial z^2}\,, \quad \varepsilon(z,u) = 1 + 4\pi\chi\left(1 - \frac{\partial u}{\partial z}\right) \,. \tag{8}$$

Since we neglect here the derivative of u over time, the same equation can be written also for the electric field $E_x = -\dot{A}/c$,

$$\frac{\varepsilon(z,u)}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial z^2}\,, \quad \varepsilon(z,u) = 1 + 4\pi\chi\left(1 - \frac{\partial u}{\partial z}\right)\,. \tag{9}$$

B. Equation for sound

We rewrite the full Lagrangian using Eq. (4) and obtain

$$\mathcal{L} = \mathcal{L}_{\rm em} + \frac{\rho}{2}(\partial_t u)^2 - \rho s^2(\partial_z u)^2 + \frac{j}{c}[A(z) + uj\partial_z A] + \mathcal{L}_j(j)\,. \tag{10}$$

The Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \frac{\partial u}{\partial t}} + \frac{\partial}{\partial z}\frac{\partial L}{\partial \frac{\partial u}{\partial z}} = \frac{\partial L}{\partial u} \tag{11}$$

yields

$$\rho(\partial_{tt}u - s^2\partial_{zz}u) = f = \frac{1}{c}j\partial_z A\,, \tag{12}$$

where f is the force density, describing action of light on vibrations. We do not intend to describe optomechanical cooling or spring effects. As such, we will evaluate f neglecting the displacement u, i.e. will assume that $j \equiv j[A(z+u)] \to j[A(z)] = \partial_t P(z)$.

Hence, equation for the force density can be rewritten as Now in the simplest approximation we get for the force density

$$\begin{aligned} f = \frac{1}{c}j\partial_z A = \frac{1}{c}\frac{\partial P}{\partial t}B = \frac{1}{c}\frac{\partial PB}{\partial t} - \frac{1}{c}P\frac{\partial B}{\partial t} = \frac{1}{c}\chi\frac{\partial EB}{\partial t} + \chi E\frac{\partial E}{\partial z} \\ = \chi\left[\frac{1}{c}\frac{\partial(EB)}{\partial t} + \frac{1}{2}\frac{\partial}{\partial z}(E^2)\right]\,, \end{aligned} \tag{13}$$

where $B_y = \partial_z A_x$ is the magnetic field. This result apparently agrees with the one in (Gordon, 1973). Two contributions correspond to light pressure and ponderomotive force.

II. PHONORITONS

Let us now consider light-sound interaction in the presence of strong optical pump $E\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}$.

$$\frac{\varepsilon}{c^2}\frac{\partial^2 E(z,t)}{\partial t^2} - \frac{\partial^2 E(z,t)}{\partial z^2} = +\frac{4\pi\chi}{c^2}\frac{\partial u}{\partial z}\frac{\partial^2 E}{\partial t^2}\,, \tag{14}$$

$$\rho\frac{\partial^2 u}{\partial t^2} - s^2\frac{\partial^2 u}{\partial z^2} = \chi\left[\frac{1}{c}\frac{\partial(EB)}{\partial t} + \frac{1}{2}\frac{\partial}{\partial z}(E^2)\right]\,, \tag{15}$$

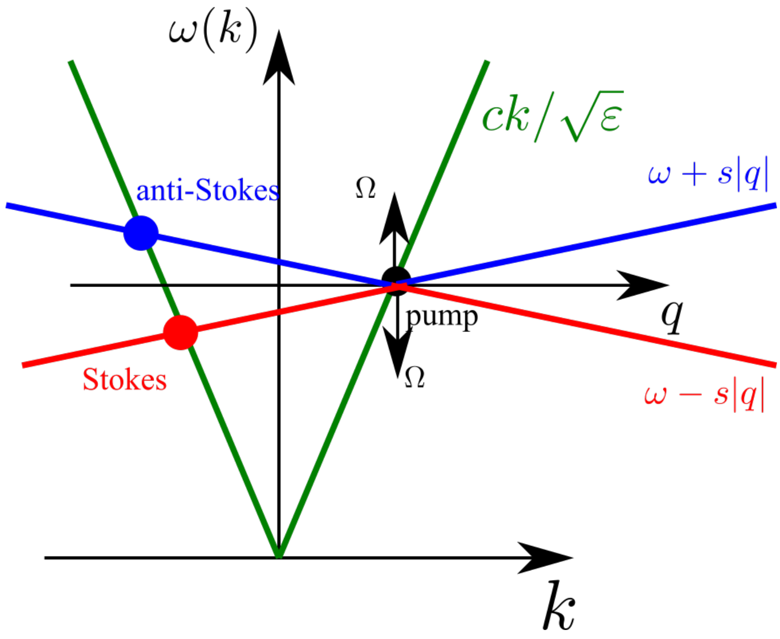

FIG. 1 Mandelstam-Brillouin scattering. Energy and momentum conservation conditions

Here χ is the polarizability of the medium and $\varepsilon = 1 + 4\pi\chi$. We write the ansatz for electric field and vibrations as

$$E(z,t) = E_0 \mathrm{e}^{\mathrm{i}kz - \mathrm{i}\omega t} + E_{\mathrm{S}} \mathrm{e}^{\mathrm{i}k_{\mathrm{S}}z - \mathrm{i}\omega_{\mathrm{S}}t} + E_{\mathrm{aS}} \mathrm{e}^{\mathrm{i}k_{\mathrm{aS}}z - \mathrm{i}\omega_{\mathrm{aS}}t} + c.c.\,, \tag{16}$$

and

$$u(z,t) = u\mathrm{e}^{\mathrm{i}qz - \mathrm{i}\Omega t} + u^* \mathrm{e}^{-\mathrm{i}qz + \mathrm{i}\Omega t}\,. \tag{17}$$

Due to the energy and momentum conservations we obtain for Stokes scattering

$$\omega_{\mathrm{S}} = \omega - \Omega, \quad k_{\mathrm{S}} = k - q \tag{18}$$

and for anti-Stokes scattering

$$\omega_{\mathrm{aS}} = \omega + \Omega, \quad k_{\mathrm{aS}} = k + q\,. \tag{19}$$

These conditions describe so-called Mandelstam-Brillouin scattering. The energy-momentum Since $s/c \sim 10^{-5} \ll 1$, we have $\Omega \ll \omega$, and $\omega_{\mathrm{S}} \approx \omega_{\mathrm{aS}} \approx \omega$. Hence, the energy and momentum conservation conditions are satisfied only in the backscattering geometry, see Fig. 1, when

$$|k| \approx |k_{\mathrm{aS}}| \approx |k_{\mathrm{S}}| \approx 2|q|, \tag{20}$$

so that

$$q \approx 2k \approx \frac{2\omega}{c}\sqrt{\varepsilon(\omega)}\,. \tag{21}$$

Hence, measurement of the Stokes shift Ω allows one to determine $q = \Omega/s = 2|k|$ assuming that the sound velocity is known. Given Eq. (21) this allows one to find the dispersion law $\omega\sqrt{\varepsilon}/c$ almost directly: one measures just the dependence of Stokes shift on incident photon frequency. Such approach has been used to map dispersion of excitonic polaritons in crystals and superlattices, see e.g. (Jusserand *et al.*, 2012).

Substituting Eq. (16) and Eq. (17) into Eq. (14) and Eq. (15) we find

$$\left(k^2 - \frac{\varepsilon\omega^2}{c^2}\right) E_0 = 0 \tag{22}$$

$$\left(k_{\rm S}^2 - \frac{\varepsilon\omega_{\rm S}^2}{c^2}\right) E_{\rm S} = 4\pi {\rm i} q\chi \frac{\omega_{\rm S}^2}{c^2} E_0 u^*\,. \tag{23}$$

$$\left(k_{\rm aS}^2 - \frac{\varepsilon\omega_{\rm aS}^2}{c^2}\right) E_{\rm aS} = -4\pi {\rm i} q\chi \frac{\omega_{\rm aS}^2}{c^2} E_0 u\,. \tag{24}$$

and

$$\rho(s^2q^2 - \Omega^2)u = {\rm i}q\chi(E_0E_{\rm S}^* + E_0^*E_{\rm aS})\,. \tag{25}$$

The light pressure term $\frac{1}{c}\frac{\partial(EB)}{\partial t}$ in the optical force has been neglected since $q \sim \Omega/s \gg \Omega/c$. Gathering all together we obtain the following dispersion equation for hybridized light and sound:

$$s^2q^2 - \Omega^2 = -\Sigma(\Omega, q), \quad \Sigma = -\delta\left(\frac{1}{(c^2k_{\rm aS}^2/\varepsilon) - \omega_{\rm aS}^2} + \frac{1}{(c^2k_{\rm S}^2/\varepsilon) - \omega_{\rm S}^2}\right), \quad \delta = \frac{4\pi q^2\omega^2\chi^2|E_0|^2}{\rho\varepsilon}\,. \tag{26}$$

Here in the expression for δ we make use of $\Omega \ll \omega$, so both $\omega_{\rm S}^2$ and $\omega_{\rm aS}^2$ are replaced by Ω^2. The solution of Eq. (26) in case when $\varepsilon(\omega)$ has an excitonic resonance are termed "phonoritons": "polaritons+phonons" (Ivanov and Keldysh, 1982).

Eq. (26) very much reminds the equation we had for light interacting with vibrations in a cavity,

$$(\Omega_m + \Sigma - \Omega - {\rm i}\gamma_m)a = 0, \quad \Sigma = -\frac{g^2|c_0|^2}{2\Omega}\left(\frac{1}{\omega_c - g\bar{x} - \omega_{\rm S} + {\rm i}\Gamma_c} + \frac{1}{\omega_c - g\bar{x} - \omega_{\rm aS} - {\rm i}\Gamma_c}\right). \tag{27}$$

Important the dispersion resulting from Eq. (26), shown in Fig. 2, is the nonreciprocity:

$$\omega_{\rm S}(k_{\rm S}) \neq \omega_{\rm S}(-k_{\rm S}), \quad \omega_{\rm aS}(k_{\rm aS}) \neq \omega_{\rm aS}(-k_{\rm aS}), \quad \Omega(q) \neq \Omega(-q)\,. \tag{28}$$

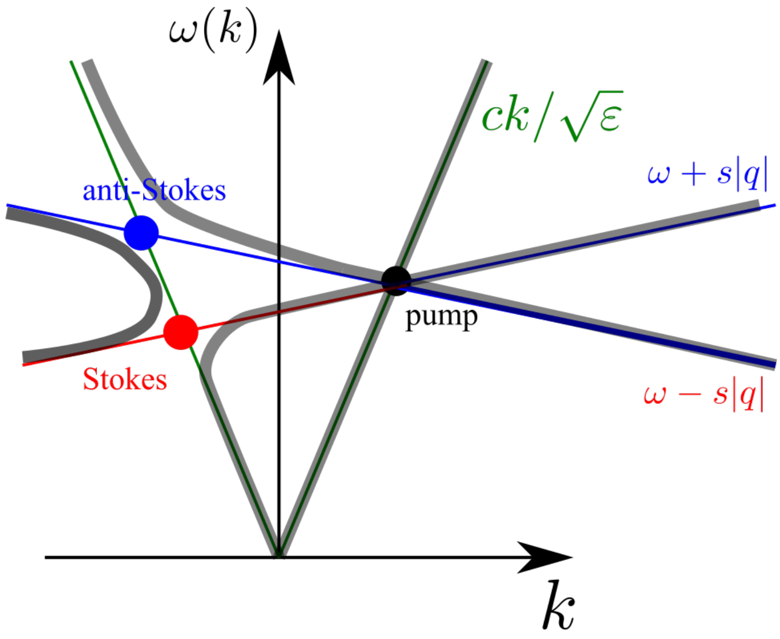

FIG. 2 Schematic illustration of the phonoriton dispersion.

Lorentz reciprocity is broken due to the presence of the pump: i.e. transmission coefficients of light or sound from left to right and from right to left are different.

Equation Eq. (26) will yield avoided crossings between light and sound dispersion, as illustrated in Fig. 2.

For anti-Stokes scattering these are "frequency-" avoided crossing, i.e. band gaps are formed for frequency. For Stokes scattering these are "wave vector-" avoided crossings, i.e. band gaps are formed for wave vectors. Physically,"wave vector-" avoided crossings mean instability, i.e. waves with certain wave vectors are amplified exponentially due to the optomechanical heating effect.

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Dynamical Casimir Effect

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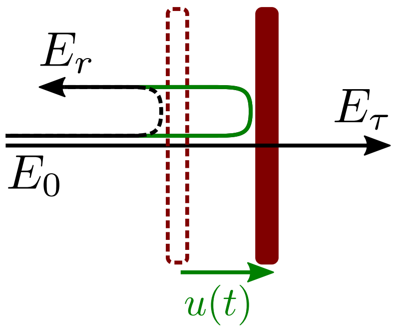
(Dated: May 6, 2022)

I. CLASSICAL REFLECTION FROM A MOVING MIRROR

Classical reflection from a moving mirror:



Consider a plane wave $E(t) = E_0 e^{-i\omega t}$ incident on a layer trembling as $u(t) = u e^{-i\Omega t} + u^* e^{i\Omega t}$

Layer displacement modifies the reflected wave phase by $\varphi(t) = 2(\omega/c)u(t)$

Reflected wave up to u-linear terms

$$E_r = rE_0 e^{-i\omega t} e^{i\varphi(t)} \approx rE_0 e^{-i\omega t} + 2i\frac{\omega}{c} rE_0 \Big[\underbrace{e^{-i(\omega+\Omega)t}u}_{\text{anti-Stokes}} + \underbrace{e^{-i(\omega-\Omega)t}u^*}_{\text{Stokes}}\Big]$$

Transmitted wave phase remains **unaffected**: $E_\tau = \tau E_0 e^{-i\omega t} + 0 + 0$

See [Poshakinskiy, ANP, PRX **9**, 011008 (2019)]

II. DYNAMICAL CASIMIR EFFECT

Vecrtor potential

$$A = \sum_k \sqrt{\frac{2\pi}{\omega}} (a_k e^{-i\omega t + ikz} + a_k^\dagger e^{i\omega t - ikz})$$

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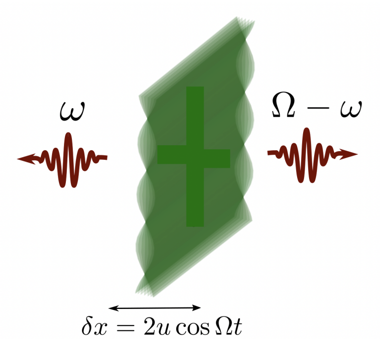

$\omega' = \Omega - \omega$

$$A_{\rm out} = \sqrt{\frac{2\pi}{\omega'}}(a_{k'}{\rm e}^{-{\rm i}\omega' t+{\rm i}k'z} + a^\dagger_{k'}{\rm e}^{{\rm i}\omega' t-{\rm i}k'z})$$

Ideal mirror, $r=-1$

$$\frac{1}{\sqrt{\Omega-\omega}}a_{k'}^{\dagger,{\rm out}} = -\frac{2{\rm i}u\omega}{c}\frac{1}{\sqrt{\omega}}a_k^{\rm in}$$

Here we took into account the fact that the outgoing wave should have $k'<0$. Hence, $k' = |\omega-\Omega| = \Omega-\omega$ $(\Omega>\omega)$. Hence, in the scattered field it is the $a^\dagger_{k'}{\rm e}^{{\rm i}\omega' t-{\rm i}k'z}$ term that corresponds to the left-going wave.

Vacuum fluctuations:

$$\langle a_k^{\rm in} a_{k'}^{{\rm in},\dagger}\rangle = \delta_{kk'}$$

Emission spectrum:

$$\langle a_{k'}^{\dagger,{\rm out}} a_{k'}^{\rm out}\rangle = 4\frac{u^2}{c^2}\omega(\Omega-\omega)$$

,

$$n(\omega) = 2\sum_{k'}\langle a_{k'}^{\dagger,{\rm out}} a_{k'}^{\rm out}\rangle \equiv 8\int_0^\Omega \frac{{\rm d}\omega}{2\pi}\frac{u^2}{c^2}\omega(\Omega-\omega) \tag{1}$$

Eq. (1) agrees with Eq. (7) in [A. Lambrecht, M.-T. Jaekel and S. Reynaud, PRL 77, **615** (1996)], see also M.-T. Jaekel and S. Reynaud, Quantum Opt. **4** 39 (1992).

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and recent video from Franko Nori

With feedback: Casimir-Rabi splitting, Macr`i et al., PRX **8**, 011031 (2018)

Lecture 5. Optomechanical interferometry

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Contents

I. STANDARD QUANTUM LIMIT

A. Particle

Let us consider a particle with coordinate x and momentum p. Due to the Planck uncertainly

$$\Delta p\Delta x \geq \frac{\hbar}{2}\,. \tag{1}$$

Suppose that we measure the coordinate twice at the time moments $t = 0$ and $t = \tau$. The first measurement has uncertainty Δx_0 and as a results introduces momentum $\Delta p = \hbar/(2\Delta x_0)$.

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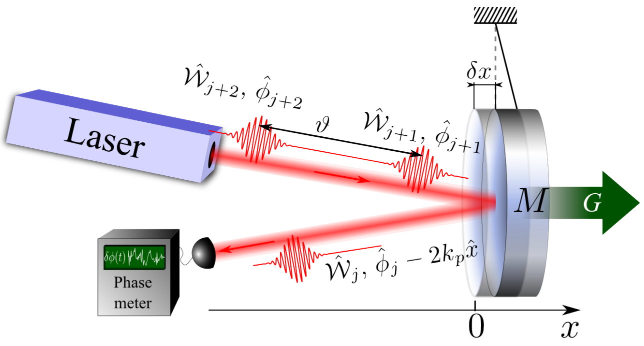

FIG. 1 Simplified scheme of optical measurement of mechanical object position in an interferometer(from Wikipedia

This momentum leads to the motion of the particle and additional uncertainty in the coordinate measurement $\Delta x_1 = \tau\Delta p/m$. Then we measure the position again and obtain $x = (x_0 + x_1)/2$ with the uncertainty

$$\Delta x^2 = \frac{1}{2}(\Delta x_0^2 + \Delta x_1^2) = \frac{1}{2}\Delta x_0^2 + \frac{\hbar^2\tau^2}{8\Delta x_0^2 m}. \tag{2}$$

This expression has a minimum at

$$\Delta x = \Delta x_0 = \sqrt{\frac{\hbar\tau}{2m}}\,, \tag{3}$$

which is a so-called standard quantum limit (Braginskiĭ, 1967).

B. Optomechanical interferometer

The same limit applies for the measurement of mirror position in a optomechanical interferometer (Caves, 1980), see Fig. 1. Suppose that laser has average power P. During the measurement time τ the number of photons incident on a mirror is $N = P\tau/\hbar\omega$ and the fluctuation of this number is $\Delta N \sim \sqrt{N}$. Hence, the precision of measurement of mirror position is

$$\Delta x_0 \propto \lambda\frac{\Delta N}{N} \sim \lambda\sqrt{\frac{\hbar\omega}{P\tau}} \tag{4}$$

where λ is the light wave length. On the other hand, the radiation pressure exerts momentum $N\hbar\omega/c$ on the mirror with the fluctuation $\Delta p \sim \sqrt{N}\hbar\omega/c$. Hence, the mirror moves by

$$\Delta x_1 = \frac{\tau}{m}\Delta p = \frac{\tau}{m}\sqrt{\frac{P\tau}{\hbar\omega}}\,. \tag{5}$$

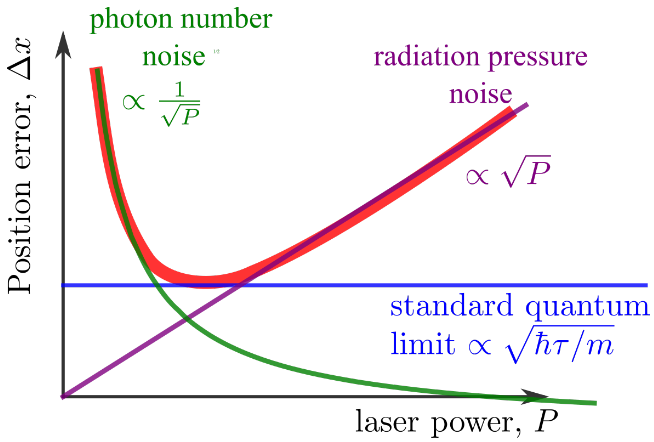

FIG. 2 Contributions to noise in optical measurement of mechanical object position

In order to achieve highest precision we need a balance between the two contributions Eq. (4) and Eq. (5), as shown in Fig. 2. The Standard Quantum Limit

$$(\Delta x^2)_{\min} \sim \frac{\hbar\tau}{m} \tag{6}$$

is reached for

$$P \sim \frac{c\lambda m}{\tau^2}. \tag{7}$$

The measurement time $\tau \sim 10^{-2}$ sec is determined by the desired gravitational waves frequency ~ 100 Hz.

II. SQUEEZED LIGHT FOR LIGO

The following summary is based on the reviews (Leuchs, 1988, 2002). Original idea to used squeezed light in gravitational interferometry was proposed in (Caves, 1981). It does not allow to surpass standard quantum limit but allows more precision for the same power. The modern version to go beyond standard quantum limit was first presented in (Unruh, 1983).

A. Beam-splitter

Let us consider for simplicity a thin 50/50 beam splitter (Fig. 4) that satisfies two equations:

$$|r|^2 = |t|^2 = \frac{1}{2} \qquad \text{(50/50 splitting without losses)} \tag{8}$$

$$t = 1 + r \qquad \text{(electric dipole resonance for a thin beamsplitter)} \tag{9}$$

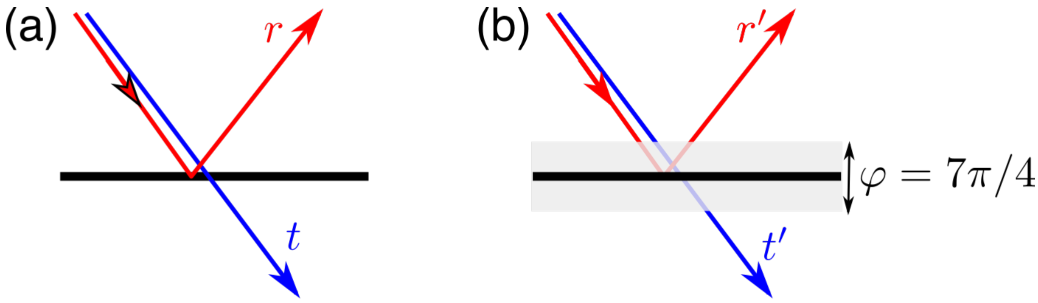

FIG. 3 Schematics of a beamsplitter, reflection (r) and transmission (t). (b) is the same but with different choice of phases, $r' = \mathrm{e}^{7\mathrm{i}\pi/4} r, t' = \mathrm{e}^{7\mathrm{i}\pi/4} t$

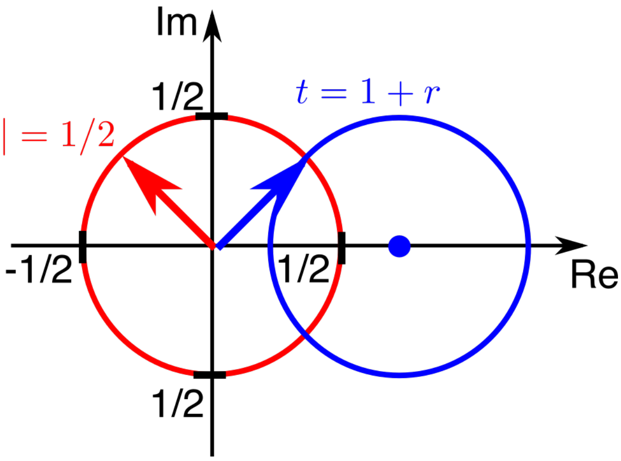

FIG. 4 Graphical solution of Eqs. (8)

The second condition is not necessary. It is easy to check that both conditions are satisfied for

$$r = \frac{\mathrm{i} - 1}{2} \equiv \frac{1}{\sqrt{2}} \mathrm{e}^{3\mathrm{i}\pi/4}, \quad t = \frac{\mathrm{i} + 1}{2} \equiv \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\pi/4}\,. \tag{10}$$

Now it is instructive to change the phases as shown in Fig. 4(b) which means that

$$r' = \mathrm{e}^{7i\pi/4} r = \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\pi/2}, \quad t' = \mathrm{e}^{7i\pi/4} t = \frac{1}{\sqrt{2}}\,. \tag{11}$$

B. Classical and quantum light

We remind that for a classical oscillator

$$\mathcal{H} = \frac{p^2 + \omega^2 q^2}{2} = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right), \quad p = \sqrt{\hbar\omega}\mathrm{i}\frac{a^\dagger - a}{\sqrt{2}}, \quad q = \sqrt{\hbar}\frac{a + a^\dagger}{\sqrt{2\omega}}, \quad [p, q] = -\mathrm{i}\hbar \tag{12}$$

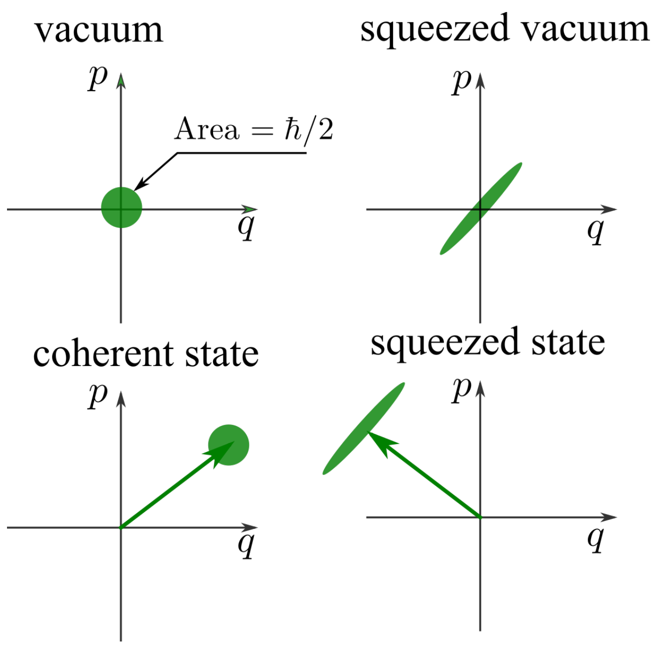

FIG. 5 Illustration of different types of classical and quantum light

For electromagnetic field the energy is defined in the same way (we disregard the polarization and dispersion) ,

$$\mathcal{H} = V\frac{E^2 + H^2}{8\pi} \tag{13}$$

and the quantization can lead to $E \sim q$, $H \sim p$ with some finite $\Delta p \Delta q \sim \Delta E \Delta H \geq \hbar/2$. Hence, we can introduce the concept of classical and squeezed light, as well as squeezed vacuum, see Fig. 5.

C. Quantum light in an interferometer

Formation of noise in the optomechanical interferometer is illustrated in Fig. 7. The amplitudes in output ports 3 and 4 are proportional to $A + C$ and $A - C$, respectively. Hence, for $C = 0$ (squeezed vacuum input), the vacuum noise does not contribute to the amplitudes (Caves, 1981). Importantly, this idea does not take into account radiation pressure noise which is determined by the difference of amplitudes in the arms. So the only impact of this idea is to achieve more sensitivity for the same power. In order to beat standard quantum limit one has also to control the *phase* of squeezing in order to compensate the radiation pressure noise, see Fig. 8. In this case the radiation pressure noise is proportional to the difference of intensities in two arms of the interferometer, i.e. it is proportional to C. This pressure leads

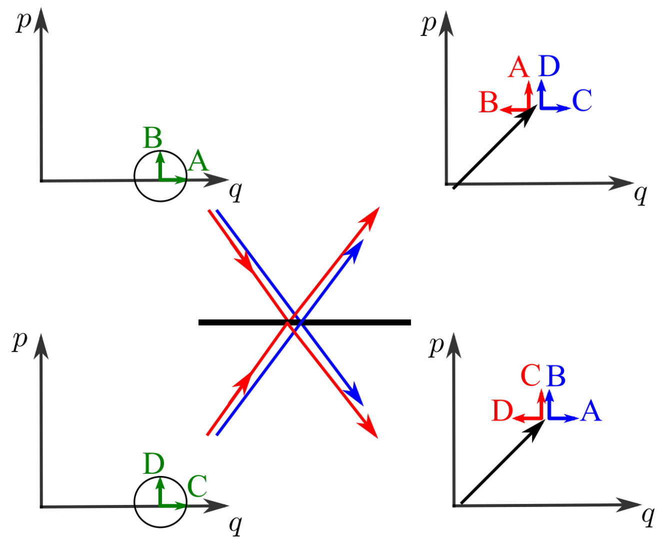

FIG. 6 Interference of two beams at a beam splitter with $r' = \mathrm{i}/\sqrt{2}$, $t' = 1/\sqrt{2}$.

to small change of phase in the upper arm, i.e. fluctuation of φ. The corresponding arrow is shown by green color at output φ, it points in the direction $3\pi/2 - \varphi$. The key point is that the projections of the photon noise (blue and maroon arrows C) and radiation pressure (green arrow C) on the photon amplitude direction cancel each other. This allows to beat standard quantum limit.

III. MORE READING

- Popular book about gravitational waves and the concept of LIGO with a lot of historical details: Kip Thorne, "Black Holes & Time Warps: Einstein's Outrageous Legacy ", W. W. Norton & Company (1994)

Quantum measurement:

- Vladimir B. Braginsky, Farid Ya Khalili, "Quantum measurement", Cambridge University Press (1995)
- C. M. Caves, "Quantum-Mechanical Radiation-Pressure Fluctuations in an Interferometer," Phys. Rev. Lett. **45** 75 (1980)
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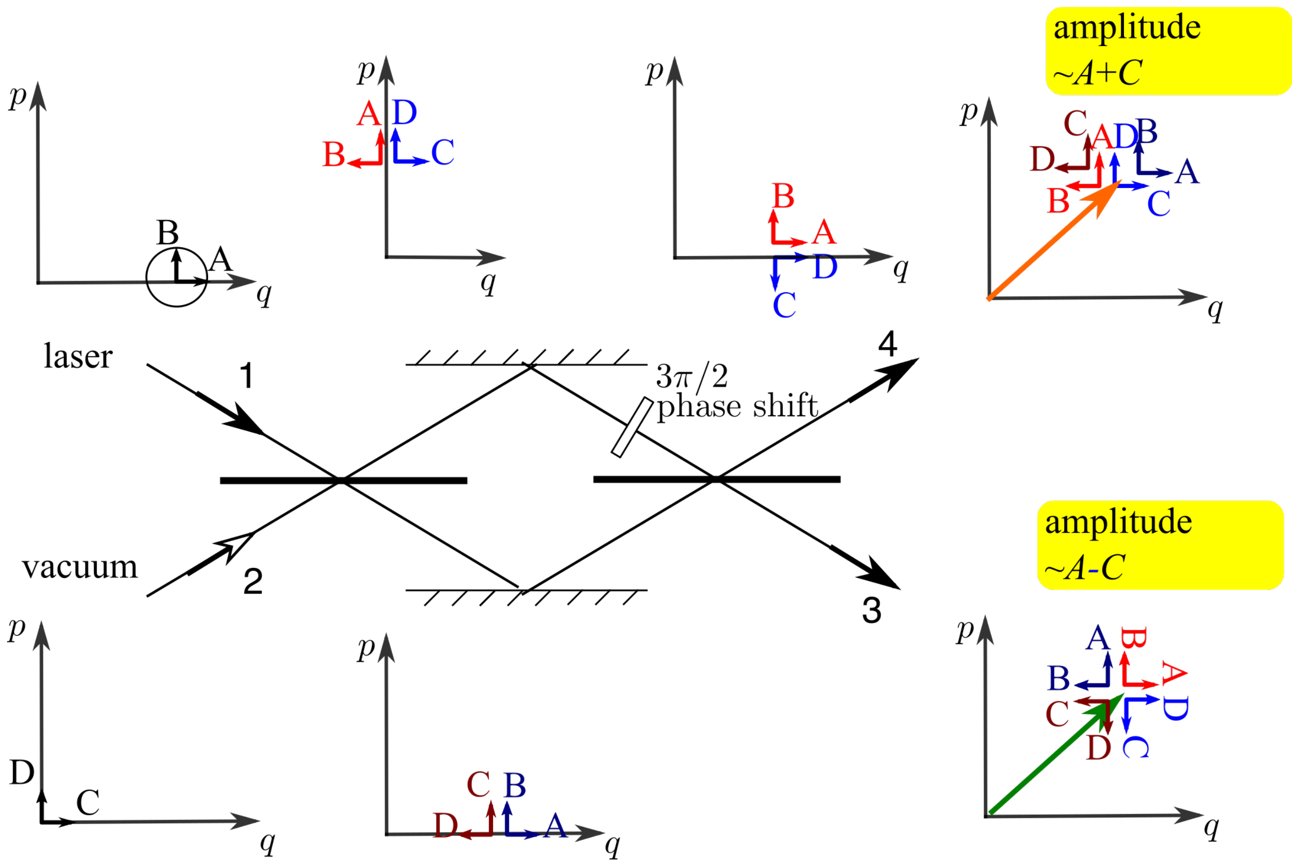

FIG. 7 (after (Leuchs, 1988))Noise in optomechanical interferometer

- W. G. Unruh, “Quantum Noise in the Interferometer Detector,” in Quantum Optics, Experimental Gravity, and Measurement Theory, Springer US, 647 (1983)
- More modern overview of theory: H. J. Kimble, Y. Levin, A. B. Matsko, K. S. Thorne, and S. P. Vyatchanin, “Conversion of conventional gravitational-wave interferometers into quantum nondemolition interferometers by modifying their input and/or output optics,” Phys. Rev. D, **65**22002 (2001)
- Modern experiments on LIGO: J. Aasi et al., “Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light,” Nature Photonics, **7**,613 (2013)
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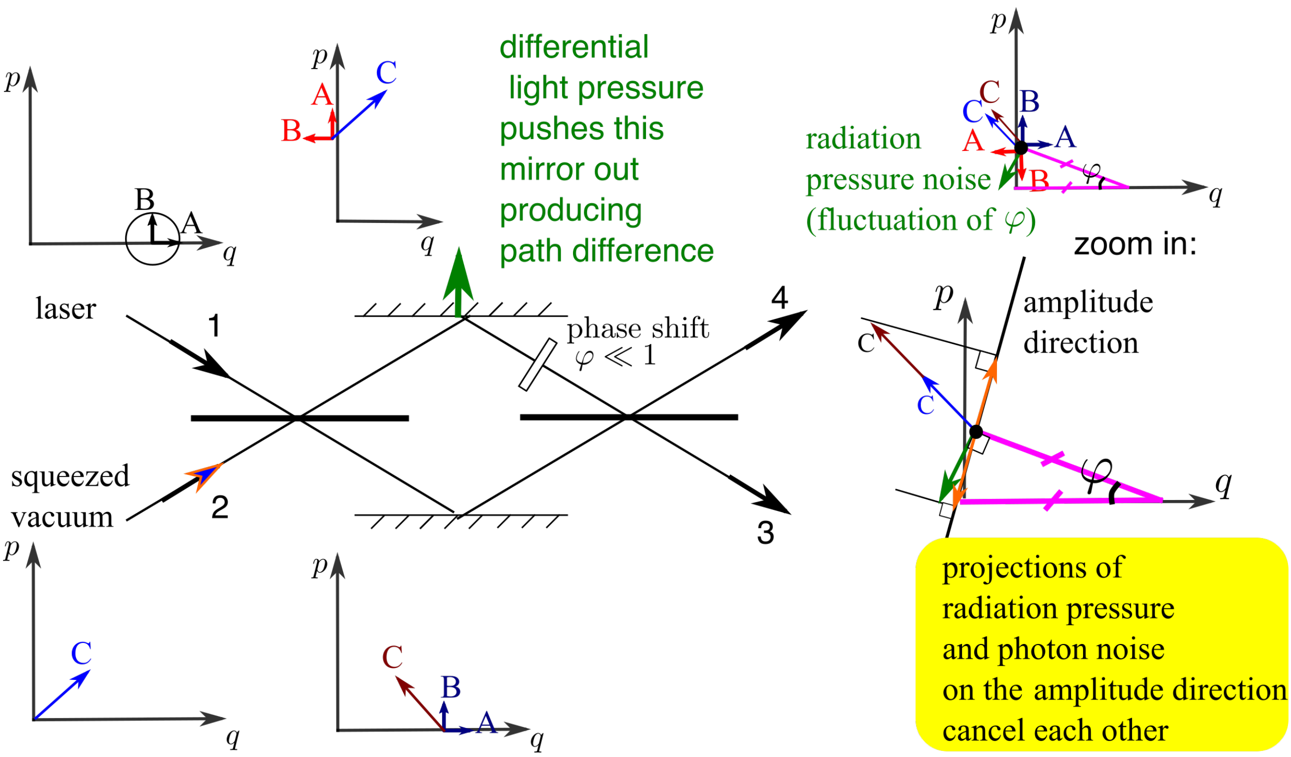

FIG. 8 (Gerd Leuchs, private communication) Illustration of squeezed vacuum input with variable squeezing phase for mutual cancellation of radiation pressure (green arrow) and photon noise (C).

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