Concepts of condensed matter physics

Spring 2016

Exercise #3

Due date: 16/05/2016

1. In this question you will re-derive the BCS theory studied in class and use it to calculate a few properties of superconductors. Our starting point is the Hamiltonian of electrons interacting via an attractive point contact interaction ($g > 0$):

$$\hat{H} = \int d^3x \left[\sum_{s=\uparrow,\downarrow} c_s^\dagger(x) \left(-\frac{\nabla^2}{2m} - \mu \right) c_s(x) - g c_\uparrow^\dagger(x) c_\downarrow^\dagger(x) c_\downarrow(x) c_\uparrow(x) \right].$$

a. Write the Hamiltonian in momentum space, and then transform it to a quadratic form by assuming the order parameter $\Delta = \frac{g}{\Omega} \sum_k c_{-k\downarrow} c_{k\uparrow}$ is weakly fluctuating (i.e., by performing mean field). Here Ω is the system's volume.

b. Diagonalize the quadratic Hamiltonian and find the spectrum of excitations.

c. What is the ground state wavefunction? What is the ground state energy? Show that taking $\Delta = 0$ we recover the known non-interacting ground state energy.

d. Using the ground state wavefunction, write a self-consistent equation ("the BSC gap equation") for Δ. Solve this equation for small values of g.

e. Extend the gap equation to finite temperatures by promoting the average with respect to the ground state to a thermal average.

f. Find the critical temperature T_c above which superconductivity is destroyed. What is the value of Δ slightly below the transition?

2. In this question you will find the spectrum of the above BCS theory in the presence of spin-orbit and Zeeman coupling. In **one-dimension**, it is given by:

$$\hat{H} = \int dx \left[\sum_{s,s'} c_s^\dagger(x) \left(-\frac{\partial_x^2}{2m} + iu\sigma_z^{ss'} \partial_x + B\sigma_x^{ss'} - \mu \right) c_{s'}(x) - g c_\uparrow^\dagger(x) c_\downarrow^\dagger(x) c_\downarrow(x) c_\uparrow(x) \right]$$

a. First, neglecting g, diagonalize the quadratic Hamiltonian by going to momentum space. Draw the spectrum (qualitatively) – how does the spin-orbit and Zeeman terms alter the parabolic spectrum of free electrons ($E = \frac{k^2}{2m} - \mu$).

b. Introducing finite g and performing a mean field approximation, write a quadratic Hamiltonian in the form:

$$H = E_0 + \sum_k \vec{\Psi}_k^\dagger h_{BDG} \vec{\Psi}_k$$

What is the form of the spinor $\vec{\Psi}$, and what is the dimension of the matrix h_{BDG}?

c. Diagonalize h_{BDG} and find the spectrum of excitations.

d. Show that by changing the ratio Δ / B, we reach a point in which the gap to excitations closes. Draw the spectrum at this point.

3. In this question you will use the Hubbard-Stratonovich transformation to re-derive the gap equation and then use general arguments to find the Ginzburg-Landau theory. We start with the expression for the partition function:

$$Z = \int D[\psi,\bar{\psi}] e^{-\int_0^\beta d\tau \int dx \left[\bar{\psi}_\sigma \left(\partial_\tau + ie\phi + \frac{1}{2m}(-i\nabla - e\mathbf{A})^2 - \mu\right)\psi_\sigma - g\bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow\right]}$$

a. Show that we can decouple the interactions using the following Hubbard-Stratonovich transformation:

$$e^{\int_0^\beta d\tau \int dx g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow} = \int D[\Delta,\bar{\Delta}] e^{-\int_0^\beta d\tau \int dx \left[\frac{|\Delta|^2}{g} - \left(\bar{\Delta}\psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow\right)\right]}.$$

b. Looking for the mean field value of Δ with $\vec{A} = \phi = 0$, show that if we look for the extremal value (assuming Δ is a constant and averaging over the fermionic part), we recover the BCS gap equation.

c. Once the action is quadratic in terms of the fermions, we can in principle integrate these out and get an effective theory for Δ. While this can in principle be done, finding all the coefficients of the expansion in the resulting theory is a formidable task. Instead, use general symmetry and gauge invariance arguments to write the lowest order terms in Δ (up to order $|\Delta|^4$ and taking only the leading term in the gauge invariant derivative). In addition, you can choose a gauge with $\phi = 0\left(\vec{A} \neq 0\right)$, and neglect the imaginary time dependence (i.e. neglect quantum fluctuations). In terms of the resulting action, what is the condition for having a superconducting state?

Write the action in terms of the order parameter $\Psi(r) = \sqrt{\left(a\frac{\rho_0 v_f^2}{T_c^2} m\right)}\Delta(r)$ (where $a = 7\zeta(3)/(24\pi^2)$, v_F is the fermi velocity and ρ_0 is the density of states) and obtain the conventional Ginzburg-Landau theory (state the dimensions of the various coefficients).

d. Bonus: By explicitly integrating out the fermions, obtain the coefficient of the term $|\Delta(r)|^2$ in the expansion.

e. By finding the extremum of this action, write a differential equation for the complex order parameter $\Psi(r)$. Identify the characteristic length-scale. In addition, write an expression for the current in the superconductor.

f. Consider a situation in which a flux Φ is inserted through a small region in the superconductor. Demanding that the current vanishes far away from this region, show that the value of Φ is strongly constrained (assume that the amplitude of Δ is a constant if we are far enough from the region containing the flux). Draw qualitatively the dependence of $\Delta(r)$ (amplitude and phase), the magnetic field $B(r)$, and the current $J(r)$ on the distance from the origin of the region (what are the characteristic length scales in each case?).