Concepts of condensed matter physics - Exercise #4

Spring 2016

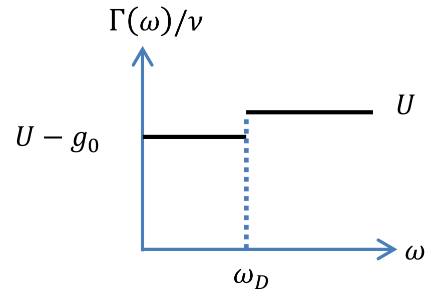
Due date: 05/06/2016

1. **Anderson-Morel gap mechanism -** The objective of this question is to resolve one of the biggest questions in conventional superconductivity. Namely, *how do two electrons which repel one another very strongly on the microscopic scale end up forming a bound-state at low energies? In class we have included only attractive interactions at a thin shell around the Fermi-level. Here we will generalize the analysis to include the fact that at high frequencies the interaction is repulsive.* To do so we will use the generalized gap equation (read the supplement to this exercise sheet. Note that here we consider a frequency dependent local interaction $g(\boldsymbol{k},\omega) = g(\omega)$):

$$\Delta(\omega) = -\int_{-\mu}^{\mu} dz \int d\xi \frac{\Gamma(\omega - z)\Delta(z)}{z^2 + \xi^2 + |\Delta(z)|^2}$$

where

$$\Gamma(\omega) = \begin{cases} \nu(U - g_0) & , |\omega| < \omega_D \\ \nu\, U & , |\omega| > \omega_D \end{cases}$$



is a frequency dependent (dimensionless) interaction (it is dimensionless because we have absorbed a density of states ν when transforming from an integration over *k* to integration over ξ). Here, U models the repulsive Coulomb interaction, and g_0 models the attractive phonon-mediated interaction that appears only for $\omega < \omega_D$ due to retardation effects (here we have chosen a notation where all couplings are positive). ξ is the fermion dispersion with constant density of states ν.

We will seek a solution for the gap which has the form

$$\Delta(\omega) = \begin{cases} \Delta_0, & \omega \ll \omega_D \\ -\Delta_1, & \omega \gg \omega_D \end{cases}$$

where $\Delta_0 > 0$ and $\Delta_1 > 0$ are to be determined from the gap equation.

a. Obtain the following equations for Δ_0 and Δ_1

$$\Delta_0 = 2\pi\nu\left[(g_0 - U)\Delta_0 \log\frac{\omega_D}{\Delta_0} + U\Delta_1 \log\frac{\mu}{\omega_D}\right]$$

$$\Delta_1 = 2\pi\nu\left[U\Delta_0 \log\frac{\omega_D}{\Delta_0} - U\Delta_1 \log\frac{\mu}{\omega_D}\right]$$

The first (second) equation is obtained by taking $\omega \ll \omega_D$ ($\omega \gg \omega_D$).

Hint: Assume $\Delta_0, \Delta_1 \ll \omega_D, \mu$, and use the logarithmic approximation

$\log(\alpha x) = \log(x) + \log(\alpha) \approx \log(x)$ for $x \gg \alpha$.

b. Solve the equations. Show that

$$\Delta_0 = \omega_D e^{-\frac{1}{g^*}}$$

what is g^*? Explain your result physically. The realistic regime is where $U \gg g_0$. Is there a solution in this regime? Discuss the limits of superconductivity.

c. Now let us obtain the same result (qualitatively) from RG. Qualitatively, to get the RG equation, we integrate out high energy configurations in the path integral formulation, thereby changing the value of the cutoff. The RG equation gives the dependence of the interaction $\tilde{g}$ on the rescaling parameter D. First, read the derivation of the RG equations in pages 115-118 of "Quantum Field Theory in Condensed Matter Physics" by Naoto Nagaosa. Then use the resulting RG equation

$$\frac{d\tilde{g}}{d\log D} = \nu\, \tilde{g}^2$$

to determine the gap (or equivalently, the critical temperature): First, write this equation in a dimensionless form using $g = \nu\tilde{g}$. Next, Integrate the equation from μ down to ω_D with a repulsive interaction $g(\mu) = g_e$. Now add a negative contribution g_{ph} and then continue the integration from ω_D down to the some cutoff $k_B T_c$ where $g(k_B T_c)$ is of order 1 and the RG equation loses its validity. The scale where the RG equations lose their validity can be interpreted as the gap. Compare this result to the previous section.

2. The XY – sine-Gordon duality and the BKT critical behavior

In this question you are asked to show the equivalence between the XY model to the sine-Gordon model:

$$S_{SG} = \frac{c}{2}\int d^2x\ (\nabla\theta)^2 - g\int d^2x\ \cos\theta.$$

Where θ is a non-compact real scalar field.

a. Expand $Z_{SG} = \int D\theta\ e^{-S_{SG}}$ in powers of g explicitly and show that it has the form

$$Z_{SG} = \sum_{n=0}^{\infty}\frac{\left(\frac{g}{2}\right)^{2n}}{(n!)^2}\prod_{j=1}^{2n}\int d^2x_j\ \langle\exp\left(i\sum_{j=1}^{2n}(-1)^j\theta(x_j)\right)\rangle$$

Where the $\langle\rangle$ brackets denote averaging with the free part $S_0 = \frac{c}{2}\int d^2x\ \ (\nabla\theta)^2$.

Hint: recall that the free part is transnationally invariant such that $\langle(\prod_{a=1}^{N} e^{i\theta(x_a)})(\prod_{b=N+1}^{N+M} e^{-i\theta(x_b)})\rangle$ is non-zero only for $N = M$.

b. Using the properties of the Gaussian average, namely

$$\langle e^A\rangle = e^{\frac{1}{2}\langle A^2\rangle}$$

for A which is a linear combination of the field θ, and the following identity

$$\langle\left(\theta(x) - \theta(x')\right)^2\rangle = \frac{C(x-x')}{c} = \frac{1}{2\pi c}\log\left|\frac{x-x'}{\xi}\right|,$$

show that the partition function may be written as follows

$$Z_{SG} = \sum_{n=0}^{\infty}\frac{\left(\frac{g}{2}\right)^{2n}}{(n!)^2}\prod_{j=1}^{2n}\int d^2x_j\ \exp\left(\frac{1}{2c}\sum_{j<i}^{2n}\sigma_i\sigma_j C(x_i - x_j)\right)$$

where σ_i denotes the sign of the vortex and ξ is a short length cutoff. This is exactly the partition function of the Coulomb gas obtained in class!

c. Repeat the derivation of the RG differential equations near the BKT transition, namely

$$\frac{dy}{dl} = xy \,; \; \frac{dx}{dl} = y^2$$

What are x and y in terms of c and g? (Here $l = \log\frac{r}{\xi}$)

d. Use the above equations to determine the screening length ξ_+ on the disordered side close to the transition. Do this by estimating the value of the running parameter l at which x and y reach order 1. Explain physically why ξ_+ is the screening length.

e. Obtain the superfluid stiffness J as a function of $t = T - T_c$ and show that it has a universal jump at T_c.

3. Superconductivity on the surface: In this question you will find that above H_{c2} there is a range of fields for which superconductivity can survive on the surface. Consult "Introduction to superconductivity", by M. Tinkham, page 135.

a. Start from the Ginzburg-Landau theory of a superconductor and neglect non-quadratic orders near the critical point. Write down the corresponding equations of motion, and using an analogy to the Schrodinger equation, find the critical field H_{c2} , above which superconductivity cannot nucleate in the interior of the sample. Write the result in terms of ϕ_0 and ξ. Can you explain the result qualitatively?

b. Consider the same physical setting with an edge at $x = 0$ (such that for $x > 0$ there is an insulator). Show that the boundary conditions take the form $\left(\frac{\nabla}{i} - \frac{2\pi A}{\phi_0}\right)\psi\Big|_n = 0$. Show that one can automatically satisfy this boundary condition by considering an auxiliary potential, containing a mirror image of the original potential in the insulating region. Does this affect the solution from part (a) well inside the superconductor (i.e., for $|x| \gg \xi$)?

c. Argue, using the auxiliary potential, that very close to the surface one can find a solution with lower energy, making the critical field higher near the surface.