Supplement to exercise 4

In this supplement we will generalize the BCS gap equation to the situation where the interactions depend on space and time. This will enable us to treat retardation effects which are an important ingredient of the phonon-mediated attractive interactions.

To analyze such spatial and temporal variations in the interactions, we write the action as

$$S = \sum_{\sigma} \int \bar{\psi}_\sigma(x) G_0^{-1} \psi_\sigma(x) - \int g(x-x') \bar{\psi}_\uparrow(x) \bar{\psi}_\downarrow(x') \psi_\downarrow(x') \psi_\uparrow(x) dx,$$

with $G_0^{-1} = \partial_\tau - \frac{\nabla^2}{2m} - \mu$. In the above, x denotes the space and time coordinates ($x = (\mathbf{x}, \tau)$). Going to k space, we find

$$S = \sum_{\sigma p} \bar{\psi}_\sigma(p) G_0^{-1}(p) \psi_\sigma(p) - \frac{1}{\sqrt{\Omega}} \sum_{p,p',q} g(p-p') \bar{\psi}_\uparrow(p) \bar{\psi}_\downarrow(-p+q) \psi_\downarrow(-p'+q) \psi_\uparrow(p').$$

Neglecting contributions from $q \neq 0$, we write the interacting part as

$$S_{int} = -\frac{1}{\sqrt{\Omega}} \sum_{p,p'} g(p-p') \bar{\psi}_\uparrow(p) \bar{\psi}_\downarrow(-p) \psi_\downarrow(-p') \psi_\uparrow(p') = -\sum_{p,p'} V_{pp'} \bar{\phi}(p) \phi(p'),$$

where we have defined $\bar{\phi}(p) = \bar{\psi}_\uparrow(p) \bar{\psi}_\downarrow(-p)$, $\phi(p') = \psi_\downarrow(-p') \psi_\uparrow(p')$, and $V_{pp'} = \frac{g(p-p')}{\sqrt{\Omega}}$. Notice that $V_{pp'}$ should be thought of as a matrix whose indices are coordinates in k-space. We now use the fact that

$$e^{\sum_{pp'} \bar{\phi}_p V_{pp'} \phi_{p'}} = \int D[\Delta, \bar{\Delta}] e^{\left(-\sum_{pp'} \bar{\Delta}_p V_{pp'}^{-1} \Delta_{p'} + \sum_p \left(\Delta_p \bar{\phi}_p + \bar{\Delta}_p \phi_p\right)\right)},$$

to write a new action:

$$S_{HS} = \sum_{\sigma p} \bar{\psi}_\sigma(p) G_0^{-1}(p) \psi_\sigma(p) + \sum_{pp'} \bar{\Delta}_p V_{pp'}^{-1} \Delta_{p'} - \sum_p \left[\Delta_p \bar{\psi}_\uparrow(p) \bar{\psi}_\downarrow(-p) + \bar{\Delta}_p \psi_\downarrow(-p) \psi_\uparrow(p)\right].$$

Deriving the mean field equations of Δ, we get

$$\sum_{p'} V_{pp'}^{-1} \Delta_{p'} - \psi_\downarrow(-p) \psi_\uparrow(p) = 0.$$

As usual, we will average over the fermionic part. To get the average of $\psi_{\downarrow}(-p)\psi_{\uparrow}(p)$, we first write the fermionic part of the action in the Nambu basis

$$\Psi_p = \begin{pmatrix} \psi_{\uparrow p} \\ \bar{\psi}_{\downarrow -p} \end{pmatrix}$$

as

$$\sum_p \bar{\Psi}_p \begin{pmatrix} G_0^{-1}(p) & -\Delta_p \\ -\bar{\Delta}_p & -G_0^{-1}(-p) \end{pmatrix} \Psi_p.$$

Since there is no coupling between different p's, we immediately write

$$\langle \psi_{\downarrow}(-p)\psi_{\uparrow}(p) \rangle = \frac{\Delta_p}{G_0^{-1}(p)G_0^{-1}(-p) + |\Delta_p|^2}.$$

Plugging everything into the mean-field equation, we have

$$\sum_{p'} V_{pp'}^{-1} \Delta_{p'} = \frac{\Delta_p}{G_0^{-1}(p)G_0^{-1}(-p) + |\Delta_p|^2}.$$

Inverting the matrix V, we get

$$\Delta_k = \sum_p V_{kp} \frac{\Delta_p}{G_0^{-1}(p)G_0^{-1}(-p) + |\Delta_p|^2}.$$

Finally, we plug in $G_0^{-1}\left[p = (\mathbf{p}, \nu)\right] = i\nu + \epsilon_{\mathbf{p}} - \mu$ and the definition of V:

$$\Delta(\mathbf{k}, \omega) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{p},\nu} \frac{g(\mathbf{k} - \mathbf{p}, \omega - \nu)\Delta(\mathbf{p}, \nu)}{(\epsilon_{\mathbf{p}} - \mu)^2 + \nu^2 + |\Delta(\mathbf{p}, \nu)|^2}.$$