Tutorial 2
The Mermin-Wagner theorem

This tutorial focuses on the famous Mermin-Wagner theorem. Basically, what the Mermin-Wagner theorem says is that 2D systems with a continuous symmetry cannot be ordered, i.e., cannot spontaneously break that symmetry. It is a very universal result that applies, for example, to magnets, solids, superfluids, and any other system characterized by a broken continuous symmetry. It illustrates the fact that as we go to lower dimensions, fluctuations become more important, and below $D = 2$, they destroy any potential ordering.

We start by focusing on a simple model: the classical xy model. In this model we have a square lattice with a planar spin on each site. The Hamiltonian takes the form

$$H = -J \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j = -J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j).$$

The system is rotationally invariant (i.e., symmetric under $\phi_i \to \phi_i + c$). However, the energy is minimal if all the spins point at the same direction, so the ground state spontaneously breaks the symmetry.

One would naively expect a ferromagnetic phase, with a broken rotational symmetry, to survive the introduction of finite temperatures (at least for low enough temperatures). This expectation is motivated by the naive intuition that the physics at zero temperature should not be different from the physics at a nearby infinitesimal temperature. At high enough temperatures, of course, there must be a transition to a disordered phase. In 3D, this is indeed the case - there is a finite temperature βJ, where β is a dimensionless number of order 1, below which the spins point at the same direction on average (even though they may be fluctuating locally).

How do we characterize order in this system? We can define a correlation function $c(\mathbf{r} - \mathbf{r}') = \left\langle e^{i(\phi(\mathbf{r}) - \phi(\mathbf{r}'))} \right\rangle$. At zero temperature, where all the spins point at the same direction this function is 1. In an ordered system, at non-zero temperatures the ϕs are homogenous on average and the correlation should remain non-zero even at large distances. This means we have long range order. On the other hand, if the system is disordered, distant spins become uncorrelated and we expect this function to go to 0 after some **finite** correlation length.

To see if our 2D system is ordered, we first assume it is and approximate the Hamiltonian based on this assumption. Then, we use the approximated Hamiltonian to calculate the correlation function. If the system is indeed ordered, self-consistency requires that the correlations stay non-zero. We will see that in 2D this is not the case, as the Mermin Wagner theorem dictates.

In the first step, we say that if the system is ordered, the fluctuations between adjacent spins are small, so we can approximate $H \approx E_0 + \frac{J}{2}\sum_{\langle i,j\rangle}(\phi_i - \phi_j)^2$. Now we have a quadratic Hamiltonian, so we can actually calculate the above correlation function. Before doing that, we make another simplification by noting that if the system is ordered, at low enough temperatures the correlation length will be much larger than the lattice spacing (which is 1 in our units). In this case, we cannot "see" the lattice, so we can go to the continuum limit (small k expansion). In doing so, we rewrite the lattice theory as a field theory with the Hamiltonian

$$H \approx \frac{J}{2}\int d^2x(\nabla\phi(\mathbf{r}))^2.$$

Note that this step is actually unnecessary, as we already had a quadratic Hamiltonian, but it will simplify later computations.

First, let us decouple the Hamiltonian. As usual, this is done by going to Fourier space, and defining

$$\phi(\mathbf{r}) = \frac{1}{2\pi}\int d^2k e^{i\mathbf{k}\cdot\mathbf{r}}\phi(\mathbf{k}).$$

Plugging this into the Hamiltonian, we get

$$H = -\frac{J}{2\left(2\pi\right)^2}\int d^2r\int d^2k\int d^2k' e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}}\phi(\mathbf{k})\phi(\mathbf{k}')\mathbf{k}\cdot\mathbf{k}'.$$

By performing the integration over $\mathbf{r}$, we get a delta-function of the form $\delta(\mathbf{k}+\mathbf{k}')$, so we have

$$H = \frac{J}{2}\int d^2k\phi(\mathbf{k})\phi(-\mathbf{k})k^2 =$$

$$= \frac{1}{2}\int d^2k\epsilon(\mathbf{k})\left|\phi(\mathbf{k})\right|^2,$$

with $\epsilon(\mathbf{k}) = Jk^2$. Note that we have used the fact that the original field $\phi(\mathbf{r})$ is real, so $\phi(-\mathbf{k}) = (\phi(\mathbf{k}))^*$. In fact, the terms for $\mathbf{k}$ and $-\mathbf{k}$ are identical, so we can actually write this as an integral over half the plane:

$$H = \int_{k>} d^2k\left|\phi(\mathbf{k})\right|^2\epsilon(\mathbf{k}).$$

Since we now have many decoupled degrees of freedom, we can immediately write

$$\langle\phi(\mathbf{k})\phi(\mathbf{k}')\rangle = \frac{\int\mathcal{D}\phi\phi(\mathbf{k})\phi(\mathbf{k}')e^{-\beta H}}{\int\mathcal{D}\phi e^{-\beta H}} = \frac{\delta(\mathbf{k}+\mathbf{k}')}{\beta\epsilon(\mathbf{k})}.$$

Now recall that we want to calculate the correlation function $c(\mathbf{r}-\mathbf{r}') = \left\langle e^{i(\phi(\mathbf{r})-\phi(\mathbf{r}'))}\right\rangle$. Since we have a Gaussian Hamiltonian, we can immediately write

$$c(\mathbf{r}-\mathbf{r}') = e^{-1/2\left\langle(\phi(\mathbf{r})-\phi(\mathbf{r}'))^2\right\rangle}.$$

To calculate the expectation value $\left\langle(\phi(\mathbf{r})-\phi(\mathbf{r}'))^2\right\rangle$, we write it in terms of the decoupled Fourier components

$$\left\langle(\phi(\mathbf{r})-\phi(\mathbf{r}'))^2\right\rangle = \int \frac{d^2k'd^2k}{(2\pi)^2}\left(e^{i\mathbf{k}\cdot\mathbf{r}}-e^{i\mathbf{k}\cdot\mathbf{r}'}\right)\left(e^{i\mathbf{k}'\cdot\mathbf{r}}-e^{i\mathbf{k}'\cdot\mathbf{r}'}\right)\langle\phi(\mathbf{k})\phi(\mathbf{k}')\rangle =$$

$$= \frac{1}{(2\pi)^2}\int d^2k'd^2k\left(e^{i\mathbf{k}\cdot\mathbf{r}}-e^{i\mathbf{k}\cdot\mathbf{r}'}\right)\left(e^{i\mathbf{k}'\cdot\mathbf{r}}-e^{i\mathbf{k}'\cdot\mathbf{r}'}\right)\frac{\delta(\mathbf{k}+\mathbf{k}')}{\beta\epsilon(\mathbf{k})} =$$

$$\frac{1}{(2\pi)^2}\int d^2k\left(e^{i\mathbf{k}\cdot\mathbf{r}}-e^{i\mathbf{k}\cdot\mathbf{r}'}\right)\left(e^{-i\mathbf{k}\cdot\mathbf{r}}-e^{-i\mathbf{k}\cdot\mathbf{r}'}\right)\frac{1}{\beta\epsilon(\mathbf{k})} = \frac{1}{2\beta\pi^2}\int d^2k\frac{(1-\cos(\delta\mathbf{r}\cdot\mathbf{k}))}{\epsilon(\mathbf{k})}.$$

Notice that in the large δr limit, we can separate the integral into the two regions $k \lesssim \delta r^{-1}$ and $k \gtrsim \delta r^{-1}$, each giving a qualitatively different contribution. In the first case we have

$$\int^{1/\delta r} d^2k\frac{(1-\cos(\delta\mathbf{r}\cdot\mathbf{k}))}{\epsilon(\mathbf{k})} \approx \int^{1/\delta r} d^2k\frac{(\delta\mathbf{r}\cdot\mathbf{k})^2}{2\epsilon(\mathbf{k})} \propto \delta r^2\int_0^{1/\delta r} dkk \to \text{const.}$$

As we will see shortly, the other term diverges, so this term will not be the leading order at large δr.

In the second case, $k \gtrsim \delta r^{-1}$, the cosine is strongly oscillating, and will again not provide leading terms, so we neglect it. We end up with the integral

$$\frac{1}{\beta J\pi}\int_{1/\delta r} dk\frac{1}{k}.$$

Note that this integral has a logarithmic divergence at high momenta. This divergence is of course an artifact of the effective continuum model, and in the original model the lattice spacing sets a high-momentum cutoff (that is, k is restricted to the Brillouin zone). We put the cutoff back by hand, and get

$$\left\langle(\phi(\mathbf{r})-\phi(\mathbf{r}'))^2\right\rangle = \frac{1}{\beta J\pi}\log(\alpha\delta r),$$

where $\alpha \propto a^{-1}$ is the cutoff. Finally, putting this back in c, we get

$$c(\mathbf{r}-\mathbf{r}') \propto (\alpha\delta r)^{-\eta(T)},$$

where $\eta = \frac{T}{2\pi J}$.

This shows that the correlation between distant spins goes to zero and the system is not ordered at **any** non-zero temperature. In particular, the physics at zero-temperature is very different from the physics at an infinitesimal temperature above it. However, the way the correlation function goes to zero is different from the behavior of disordered systems. The power law correlations show a decay without a length-scale. The correlation length is actually infinite, similar to a second order phase transition. The difference is that here we are not at an isolated point in parameter space, but find this behavior for a region of parameters. We call such a phase a quasi-long-range-ordered phase.

One may think that this result is specific to the classical xy model, but it is actually quite universal. Any classical system in 2D with a continuous symmetry will have a massless field by the Goldstone theorem. The fluctuations created by these Goldstone modes destroy the order in a similar fashion to what we have seen above - even if the corresponding Hamiltonians are much more complicated. This has been proven in very general scenarios over the years.

For example, we can study the stability of 2D solids: let's look at the case of a square lattice, and assign a displacement vector to each lattice point $\mathbf{u}_i$. Approximating the deviations of the potential from equilibrium to be harmonic, we write the energy in the form

$$\frac{K}{2}\sum_{\langle i,j\rangle}(\mathbf{u}_i-\mathbf{u}_j)^2.$$

Note the similarity of this to the form we wrote for the xy model. We can therefore immediately say that this Hamiltonian will result in the fluctuation of the form

$$\left\langle(\mathbf{u}_i-\mathbf{u}_j)^2\right\rangle \propto T\log|i-j|\,.$$

This means that the relative displacement vector between two distant sites is wildly fluctuating, and the original crystal structure is unstable.

The Mermin-Wagner theorem is not special to 2D classical problems. It actually applies to various quantum problems as well. We have seen in the previous tutorial from the path integral formulation that the partition function of a quantum many body system takes the form

$$Z=\int D[\psi,\bar{\psi}]e^{-\int_0^\beta d\tau d^d x\left(\bar{\psi}\partial\tau\psi+H[\bar{\psi},\psi]-\mu N[\bar{\psi},\psi]\right)}.$$

Thinking about the Lagrangian density as an effective classical Hamiltonian density, and about the τ (time) direction as another spatial direction, we see that this partition function describes a classical d+1 dimensional system which is finite in one direction (the time direction), and infinite in the d other directions. At zero-temperature, the system is infinite in the τ direction as well, so the quantum many-body problem is mapped into an infinite classical $D=d+1$ dimensional system. This mapping is called the quantum-classical mapping.

This way, a zero-temperature quantum problem in 1D is mapped into a 2D classical problem, where the Mermin Wagner theorem applies. This means that 1D quantum problems with a continuous symmetry cannot be ordered. A 2D quantum problem at zero-temperature is mapped onto a 3D classical problem, where order can occur. However, at finite temperatures, the system is a "thick" 2D classical system, where the Mermin-Wagner theorem should apply (if we look at large enough distances).

One last note: we have shown that the low temperature phase of the 2D xy model is quasi-long-range-ordered. It is interesting to ask whether at high temperatures a phase transition occurs between the quasi-long-range-ordered to a disordered phase. We usually associate a phase transition with a process of symmetry breaking, but here neither of the phases breaks any symmetry, so one naively expects not to find a transition. As it turns out, there is a transition, and it is called the Berezinskii-Kosterlitz-Thouless transition. Historically, it was the first example of a topological phase transition. You will study this transition extensively later in this course.