Tutorial 3
The BCS Theory of Superconductivity

In the lectures you saw a phenomenological analysis of superconductors. In particular, you saw that given some empirical results, many additional predictions can be made using the Ginzburg-Landau formalism. Historically, this approach has been very successful.

However, the theory is still incomplete without a microscopic explanation. In this tutorial we will fill this gap by reviewing the famous BCS theory, established by Bardeen, Cooper, and Schrieffer about 50 years after the initial discovery of superconductivity. Then, we will connect the microscopic picture to the phenomenological one by deriving the Ginzburg-Landau theory.

Part I
Preliminaries

The BCS theory is based on two important insights:

1. Cooper's realization that attractive interactions between electrons in the vicinity of the Fermi-energy favor the formation of bound states made of two electrons, called cooper pairs.

2. The result that interaction between two electrons, mediated by phonons, can be attractive.

Once one realizes these things, the next step is to assume that the ground state of a many body system with attractive interactions can be described in terms of a condensate of such weakly interacting pairs. The pairs satisfy Bose statistics, giving rise to a physics similar to that of a superfluid, yet different due to the fact that the bosons are now charged. We will see that this picture is capable of explaining superconductivity.

First let us elaborate on the above two crucial points:

To see that pairs of electrons can form bound states, we examine the following toy model. We imagine a filled Fermi sea. On top of that, we add two electrons which have an attractive interaction only with each other (note that they do feel the Fermi sea via the Pauli-principle). We would like to find the corresponding two-Fermion eigenstate. We assume that the total momentum is zero and that the spin-part of the wavefunction is antisymmetric. Then, we write the wavefunction as

$$\psi(\mathbf{r_1},\mathbf{r_2}) = \sum_{\mathbf{k}} \left(g_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{r_1}-\mathbf{r_2})} \right) \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right). \tag{1}$$

The Schrodinger equation takes the form

$$\left[H_0(\mathbf{r_1}) + H_0(\mathbf{r}_2) + V(\mathbf{r_1} - \mathbf{r_2}) \right] \psi(\mathbf{r_1},\mathbf{r_2}) = E\psi(\mathbf{r_1},\mathbf{r_2}), \tag{2}$$

Plugging Eq. 1 in, we get

$$\sum_{\mathbf{k}} g_{\mathbf{k}} \left[H_0(\mathbf{r_1}) + H_0(\mathbf{r}_2) + V(\mathbf{r_1} - \mathbf{r_2})\right] e^{i\mathbf{k}(\mathbf{r_1}-\mathbf{r_2})} = E \sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{r_1}-\mathbf{r_2})}.$$

In a translation invariant system we get

$$\sum_{\mathbf{k}} g_{\mathbf{k}} \left[2\epsilon_{\mathbf{k}} + V(\mathbf{r_1} - \mathbf{r_2})\right] e^{i\mathbf{k}(\mathbf{r_1}-\mathbf{r_2})} = E \sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{r_1}-\mathbf{r_2})}.$$

Multiplying by $e^{-i\mathbf{q}(\mathbf{r_1}-\mathbf{r_2})}$ and integrating over space, we get

$$2\epsilon_{\mathbf{q}} g_{\mathbf{q}} \Omega + \sum_{\mathbf{k}} \int V(\mathbf{r_1} - \mathbf{r_2}) e^{i(\mathbf{k}-\mathbf{q})(\mathbf{r_1}-\mathbf{r_2})} g_{\mathbf{k}} = E g_{\mathbf{q}} \Omega,$$

which can be written in the form

$$\sum_{\mathbf{k}} V_{\mathbf{k},\mathbf{q}} g_{\mathbf{k}} = \left(E - 2\epsilon_{\mathbf{q}}\right) g_{\mathbf{q}}, \tag{3}$$

with

$$V_{\mathbf{k},\mathbf{q}} = \frac{1}{\Omega} \int V(\mathbf{r}) e^{i(\mathbf{k}-\mathbf{q})\mathbf{r}}.$$

Obviously the energies depend on the form of the interaction V, but the phenomena we want to see should be universal for Fermions with attractive interactions, so we pick the simplest form we can think of:

$$V_{\mathbf{k},\mathbf{q}} = \begin{cases} -V & \text{If } E_F < \epsilon_{\mathbf{k}} < E_F + \Delta E \text{ and the same for } \epsilon_{\mathbf{q}} \\ 0 & \text{Otherwise} \end{cases}. \tag{4}$$

Plugging this in Eq. 3, we have

$$-V \sum_{\mathbf{k}} g_{\mathbf{k}} = \left(E - 2\epsilon_{\mathbf{q}}\right) g_{\mathbf{q}},$$

where the sum over $\mathbf{k}$ is restricted by the requirements given by Eq. 4. Dividing by $E - 2\epsilon_{\mathbf{q}}$ and summing of over $\mathbf{q}$ (with the same restrictions), we get

$$-\sum_{\mathbf{q}} \frac{V}{E - 2\epsilon_{\mathbf{q}}} = 1.$$

Transforming this into an integration over energy:

$$-\int_{E_F}^{E_F+\Delta E} d\epsilon \frac{V n(\epsilon)}{E - 2\epsilon} = 1.$$

We Integrate over a thin shell, so we assume the DOS is constant over this region, and we write

$$-V n(E_F) \int_{E_F}^{E_F+\Delta E} d\epsilon \frac{1}{E - 2\epsilon} = 1.$$

This leads to the equation

$$\frac{V n(E_F)}{2} \log\left(\frac{E - 2\left(E_F + \Delta E\right)}{E - 2E_F}\right) = 1,$$

whose solution is given by

$$E = 2E_F - 2\Delta E e^{-\frac{2}{Vn(E_F)}}.$$

So we get a state with a lower energy than that of two non-interacting electrons added exactly at the Fermi-surface. In addition, by studying the corresponding wavefunction, one can show that this is indeed a bound state. This result demonstrates a general principle: if there is an attractive interaction (which can be arbitrarily small) between the electrons, there is an instability towards the formation of pairs. One can then assume that the ground state of a many-body system with attractive interactions is composed of many weakly interacting pairs.

We now turn to study the possible origin of such attractive interactions. As it turns out, these can originate from an electron-electron interaction, mediated by phonons. We will only discuss a very qualitative picture here, but this can be made more rigorous. The idea is that an electron can pass at some time near an ion and attract it. The electron passes after a short time $\sim E_F^{-1}$, and now there is a large concentration of positive charges around the electron's original position. Using the fact that the ion can return to equilibrium only after a time $\sim \omega_D^{-1}$, which is much larger than E_F^{-1}, we find that long after the original electron has passed, there is still a concentration of positive charges. This attracts other electrons. The net effect is an attractive interaction between the two electrons (which in reality is mediated by the phonons).

Part II

BCS theory

Having the above physics in mind, we postulate that as the system becomes superconducting, there is an instability toward condensation of pairs. To investigate the physics that arises from that, we assume that the ground state of a system with attractive interactions $|\Omega_s\rangle$ is characterized by a macroscopic number of pairs. This means that $\Delta = \frac{g}{\Omega}\sum_{\mathbf{k}} \langle\Omega_s|\, \psi_{-\mathbf{k},\downarrow}\psi_{\mathbf{k}\uparrow}\,|\Omega_s\rangle$, and its complex conjugate $\bar{\Delta} = \frac{g}{\Omega}\sum_{\mathbf{k}} \langle\Omega_s|\, \psi^\dagger_{\mathbf{k}\uparrow}\psi^\dagger_{-\mathbf{k},\downarrow}\,|\Omega_s\rangle$ is non-zero. We regard these quantities as the order parameters of our system.

Using the above assumption, we use the usual mean field formulation to transform the interacting Hamiltonian into a quadratic one, neglecting some quantum fluctuations.

We start from a system of fermions with attractive interactions

$$H = \sum_{\mathbf{k},\sigma} n_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}} - \mu) - \frac{g}{\Omega}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \psi^\dagger_{\mathbf{k}+\mathbf{q}\uparrow}\psi^\dagger_{-\mathbf{k}\downarrow}\psi_{-\mathbf{k}'+\mathbf{q}\downarrow}\psi_{\mathbf{k}'\uparrow}.$$

Under our mean-field assumption, $\sum_{\mathbf{k}'} \psi_{-\mathbf{k}'+\mathbf{q}\downarrow}\psi_{\mathbf{k}'\uparrow}$, is governed by small $\mathbf{q}$'s so we take $\mathbf{q} = \mathbf{0}$, and we write

$$\sum_{\mathbf{k}'} \psi_{-\mathbf{k}'\downarrow}\psi_{\mathbf{k}'\uparrow} = \frac{\Omega\Delta}{g} + \underbrace{\sum_{\mathbf{k}'} \psi_{-\mathbf{k}'\downarrow}\psi_{\mathbf{k}'\uparrow} - \frac{\Omega\Delta}{g}}_{\equiv \frac{\Omega\delta}{g}\ \text{(Small)}},$$

and in the same way

$$\sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}\uparrow}\psi^\dagger_{-\mathbf{k}\downarrow} = \frac{\Omega\bar{\Delta}}{g} + \underbrace{\sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}\uparrow}\psi^\dagger_{-\mathbf{k}\downarrow} - \frac{\Omega\bar{\Delta}}{g}}_{\equiv \frac{\Omega\bar{\delta}}{g}\ \text{(Small)}}.$$

The interactions takes the form

$$-\frac{g}{\Omega}\sum_{\mathbf{k},\mathbf{k}'}\psi^{\dagger}_{\mathbf{k}\uparrow}\psi^{\dagger}_{-\mathbf{k}\downarrow}\psi_{-\mathbf{k}'\downarrow}\psi_{\mathbf{k}'\uparrow}=-\frac{\Omega}{g}\left(\Delta+\delta\right)\left(\bar{\Delta}+\bar{\delta}\right)\approx\frac{\Omega}{g}\left(\left|\Delta\right|^{2}+\delta\bar{\Delta}+\bar{\delta}\Delta\right).$$

Plugging this in the Hamiltonian, we get

$$H=\sum_{\mathbf{k},\sigma}n_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}}-\mu)+\frac{\Omega}{g}\left|\Delta\right|^{2}-\Delta\sum_{\mathbf{k}}\psi^{\dagger}_{\mathbf{k}\uparrow}\psi^{\dagger}_{-\mathbf{k}\downarrow}-\bar{\Delta}\sum_{\mathbf{k}}\psi_{-\mathbf{k}\downarrow}\psi_{\mathbf{k}\uparrow}.$$

This is sometimes called the Bogoliubov de-Gennes (BDG) Hamiltonian. We have transformed our interacting Hamiltonian into a quadratic mean-field Hamiltonian that captures the correct ordering in our system. Note, however, that this form is dramatically different than the type of Mean field Hamiltonians we usually write as it doesn't conserve the number of particles. The number of particles is indeed not conserved, but the parity of that number (i.e., the number of particles mod 2) remains a good quantum number.

We would like to diagonalize the BDG Hamiltonian. To do so, we define the spinor $\Psi_{\mathbf{k}}=\left(\begin{array}{cc}\psi_{\mathbf{k}\uparrow} & \psi^{\dagger}_{-\mathbf{k}\downarrow}\end{array}\right)^{T}$, in terms of which we can write

$$H=\frac{\Omega}{g}\left|\Delta\right|^{2}+\sum_{\mathbf{k}}\left(\epsilon_{\mathbf{k}}-\mu\right)+\sum_{\mathbf{k}}\Psi^{\dagger}_{\mathbf{k}}h_{BDG}\Psi_{\mathbf{k}},$$

with

$$h_{BDG}=\left(\begin{array}{cc}\epsilon_{\mathbf{k}}-\mu & -\Delta\\ -\bar{\Delta} & -\left(\epsilon_{\mathbf{k}}-\mu\right)\end{array}\right).$$

To see that this is true, let us plug the definition of $\Psi_{\mathbf{k}}$ in:

$$\begin{aligned}H &= \frac{\Omega}{g}\left|\Delta\right|^{2}+\sum_{\mathbf{k}}\left(\epsilon_{\mathbf{k}}-\mu\right)+\sum_{\mathbf{k}}\Psi^{\dagger}_{\mathbf{k}}h_{BDG}\Psi_{\mathbf{k}}=,\\ &= \frac{\Omega}{g}\left|\Delta\right|^{2}+\sum_{\mathbf{k}}\left(\epsilon_{\mathbf{k}}-\mu\right)+\sum_{\mathbf{k}}\Big[\left(\epsilon_{\mathbf{k}}-\mu\right)\left(\psi^{\dagger}_{\mathbf{k}\uparrow}\psi_{\mathbf{k}\uparrow}-\psi_{-\mathbf{k}\downarrow}\psi^{\dagger}_{-\mathbf{k}\downarrow}\right)-\left(\Delta\psi^{\dagger}_{\mathbf{k}\uparrow}\psi^{\dagger}_{-\mathbf{k}\downarrow}+\bar{\Delta}\psi_{-\mathbf{k}\downarrow}\psi_{\mathbf{k}\uparrow}\right)\Big]=\\ &= \sum_{\mathbf{k},\sigma}n_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}}-\mu)+\frac{\Omega}{g}\left|\Delta\right|^{2}-\Delta\sum_{\mathbf{k}}\psi^{\dagger}_{\mathbf{k}\uparrow}\psi^{\dagger}_{-\mathbf{k}\downarrow}-\bar{\Delta}\sum_{\mathbf{k}}\psi_{-\mathbf{k}\downarrow}\psi_{\mathbf{k}\uparrow}.\end{aligned}$$

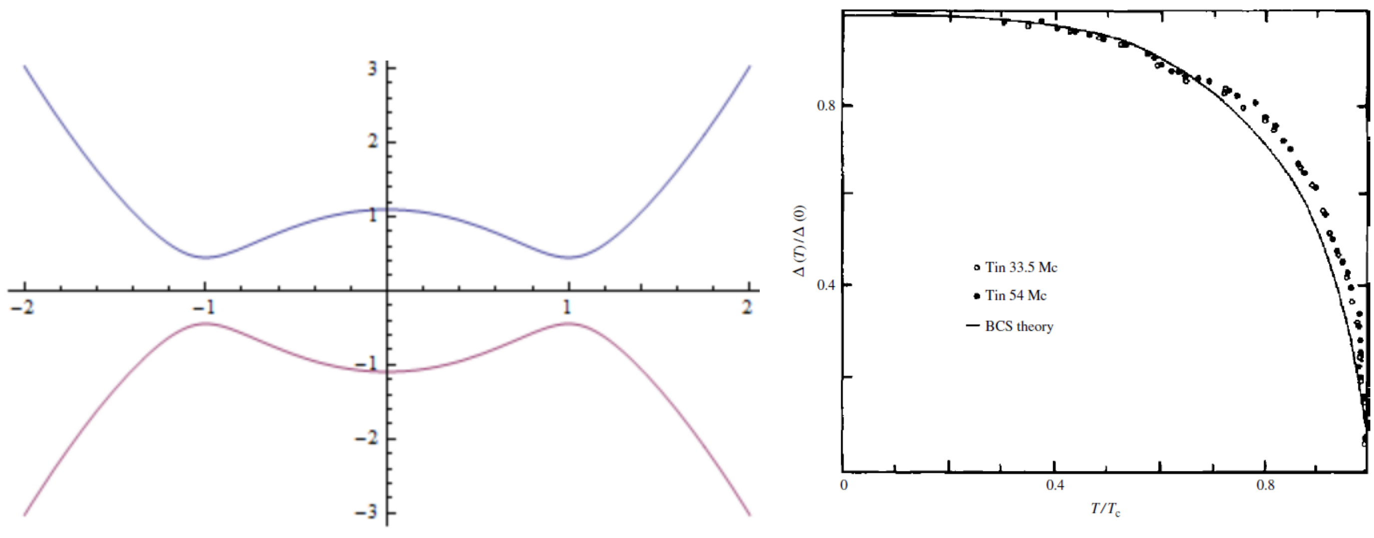
Because the matrix h_{BDG} is Hermitian, we can always perform a unitary transformation and diagonalize it, such that (assuming Δ is real)

$$Uh_{BDG}U^{-1}=\left(\begin{array}{cc}\lambda_{\mathbf{k}} & 0\\ 0 & -\lambda_{\mathbf{k}}\end{array}\right)$$

$$\chi_{\mathbf{k}}=\left(\begin{array}{c}c_{\mathbf{k},1}\\ c_{\mathbf{k},2}\end{array}\right)=U\left(\begin{array}{c}\psi_{\mathbf{k}\uparrow}\\ \psi^{\dagger}_{-\mathbf{k}\downarrow}\end{array}\right).$$

The unitary transformation can be parametrized by

$$U=\left(\begin{array}{cc}\cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}}\\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}}\end{array}\right),$$



(a) (b)

Figure 1

where $\tan(2\theta_{\mathbf{k}}) = -\frac{\Delta}{\epsilon_{\mathbf{k}}-\mu}$, and the eigenvalues are $\lambda_{\mathbf{k}} = \sqrt{\Delta^2 + (\epsilon_{\mathbf{k}} - \mu)^2}$. In terms of these, the Hamiltonian takes the diagonal form

$$H = \frac{\Omega}{g}|\Delta|^2 + \sum_{\mathbf{k}}(\epsilon_{\mathbf{k}} - \mu) + \sum_{\mathbf{k}}\left(\lambda_{\mathbf{k}} c^\dagger_{\mathbf{k},1} c_{\mathbf{k},1} - \lambda_{\mathbf{k}} c^\dagger_{\mathbf{k},2} c_{\mathbf{k},2}\right). \quad (5)$$

Taking $\epsilon_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m}$, we get the dispersion shown in Fig. 1a.

It is now simple to identify the ground state: it is the state in which all the negative energy states are occupied and the positive energy states are empty, that is

$$|\Omega_s\rangle = \prod_{\mathbf{k}} c^\dagger_{\mathbf{k},2} c_{\mathbf{k},\mathbf{1}} |0\rangle \propto \prod_{\mathbf{k}} \left(\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} \psi^\dagger_{\mathbf{k}\uparrow} \psi^\dagger_{-\mathbf{k}\downarrow}\right)|0\rangle\,,$$

where $|0\rangle$ is the vacuum of our Fock space.

To get excited states we can either destroy a $c_{\mathbf{k}2}$ particle, or create a $c_{\mathbf{k}1}$ - both with an energy cost of $\lambda_{\mathbf{k}}$. The crucial point is that there is a gap Δ to excitations. This gap is essential for superconductivity.

Recall that Δ was defined as the expectation value $\Delta = \frac{g}{\Omega}\sum_{\mathbf{k}} \langle\Omega_s|\,\psi_{-\mathbf{k},\downarrow}\psi_{\mathbf{k}\uparrow}\,|\Omega_s\rangle$. We are now in a position to write a self-consistent equation for it. All we need to do is to write the $\psi' s$ in terms of the c's, and find $\langle\Omega_s|\,\psi_{-\mathbf{k},\downarrow}\psi_{\mathbf{k}\uparrow}\,|\Omega_s\rangle = -\frac{1}{2}\sin(2\theta_{\mathbf{k}}) = \frac{\Delta}{2\lambda_{\mathbf{k}}}$. So we get the self consistent equation

$$\Delta = \frac{g}{2\Omega}\sum_{\mathbf{k}}\frac{\Delta}{\lambda_{\mathbf{k}}}.$$

Transforming this into an integral, and recalling that the attractive interaction occurs only at a thin shell of order ω_D around the Fermi-energy, we write

$$1 = \frac{g}{2}\int_{-\omega_D}^{\omega_D} d\xi \frac{n(\xi)}{\sqrt{\Delta^2+\xi^2}} \approx \frac{gn}{2}\int_{-\omega_D}^{\omega_D} \frac{d\xi}{\sqrt{\Delta^2+\xi^2}} = gn \cdot \sinh^{-1}\left(\frac{\omega_D}{\Delta}\right).$$

Solving this for Δ, and assuming the interaction is small, we get

$$\Delta \approx 2\omega_D e^{-\frac{1}{gn}}.$$

It is instructive to find the critical temperature from this formalism. To do this we need to write the self-consistency equation at finite temperatures. You will do this in the homework exercise.

To summarize this part, we now have a microscopic theory that explains the condensation of pairs and the emerging gap to excitations. However, this picture doesn't actually allow us to find the electromagnetic response of the system. To capture this part, we need to include an additional degree of freedom in our picture: the Goldstone mode associated with changing the phase of Δ. Such a treatment necessarily goes beyond the above mean field treatment, which treats Δ as a constant. This will be done next.

Part III

Deriving the Ginzburg-Landau theory

To make contact with the phenomenological analysis, and include the phase mode in the analysis, we turn to derive the Ginzburg-Landau functional from the microscopics using the Hubbard-Stratonovich transformation. This is very similar in spirit to what we already saw when we discussed magnetism.

The partition function of the system is given by

$$Z = \int D[\psi,\bar{\psi}] e^{-\int_0^\beta d\tau \int dx \left[\bar{\psi}_\sigma\left(\partial_\tau + ie\phi + \frac{1}{2m}(-i\nabla - e\mathbf{A})^2 - \mu\right)\psi_\sigma - g\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow\right]},$$

where we have introduced coupling to the electromagnetic field in the form of the minimal coupling ($\partial_\tau \to \partial_\tau + ie\phi, -i\nabla \to -i\nabla - e\mathbf{A}$).

To get the Ginzburg-Landau theory, we decouple the interacting term using

$$e^{\int_0^\beta d\tau \int dx g\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow} = \int D[\Delta,\bar{\Delta}] e^{-\int_0^\beta d\tau \int dx \left[\frac{|\Delta|^2}{g} - \left(\bar{\Delta}\psi_\downarrow\psi_\uparrow + \Delta\bar{\psi}_\uparrow\bar{\psi}_\downarrow\right)\right]}.$$

The resulting action is identical to the mean-field action we had in the previous section if we treat Δ as a constant field, giving Δ the interpretation of the superconducting order parameter we had before. However, now it's a dynamical field, and in particular, it has a phase which can fluctuate.

Defining the Nambu-spinor as

$$\Psi = \begin{pmatrix} \psi_\uparrow \\ \bar{\psi}_\downarrow \end{pmatrix}, \bar{\Psi} = \begin{pmatrix} \bar{\psi}_\uparrow & \psi_\downarrow \end{pmatrix},$$

the full action takes the form

$$Z = \int D[\psi,\bar{\psi}] D[\Delta,\bar{\Delta}] e^{-\int_0^\beta d\tau \int dx \left[\frac{|\Delta|^2}{g} - \bar{\Psi}\mathcal{G}^{-1}\Psi\right]},$$

with

$$\mathcal{G}^{-1} = \begin{pmatrix} \left[G^{(p)}\right]^{-1} & \Delta \\ \bar{\Delta} & \left[G^{(h)}\right]^{-1} \end{pmatrix},$$

and the differential operators

$$\left[G^{(p)}\right]^{-1} = -\partial_\tau - ie\phi - \frac{1}{2m}\left(-i\nabla - e\mathbf{A}\right)^2 + \mu$$

$$\left[G^{(h)}\right]^{-1} = -\partial_\tau + ie\phi + \frac{1}{2m}\left(i\nabla - e\mathbf{A}\right)^2 - \mu.$$

We want an effective action for the order parameter Δ, so we would like to integrate out the Grassmann fields. This is simple, and the result is

$$Z = \int D[\Delta,\bar{\Delta}] e^{-\int_0^\beta d\tau \int dx \left[\frac{|\Delta|^2}{g}\right] + \log\det\mathcal{G}^{-1}}.$$

If we want to recover the mean field results we can derive the equations of motion out of the effective action, neglecting quantum fluctuations in Δ, and get exactly the same gap equation we got in our mean field analysis above.

But we want to go beyond that, and consider the effect of fluctuations. We will assume that Δ is small, which is true close to the transition, and expand $\log\det\mathcal{G}^{-1} = tr\log\mathcal{G}^{-1}$ to lowest orders.

To do so, we write $\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + \hat{\Delta} = \mathcal{G}_0^{-1}\left(1 + \mathcal{G}_0\hat{\Delta}\right)$, with $\mathcal{G}_0^{-1} = \mathcal{G}^{-1}(\Delta = 0)$, and $\hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{pmatrix}$, such that

$$tr\log\mathcal{G}^{-1} = tr\log\mathcal{G}_0^{-1} + tr\log\left(1 + \mathcal{G}_0\hat{\Delta}\right) = tr\log\mathcal{G}_0^{-1} - \sum_{n=0}^{\infty}\frac{1}{2n} tr\left(\mathcal{G}_0\hat{\Delta}\right)^{2n}.$$

We will not calculate the traces here, but those who are interested in such details are referred to Altland & Simons, chapter 6. The results are:

$$\begin{aligned} S_{GL} &= \beta\int dx\left[\frac{r}{2}|\Delta|^2 + \frac{c}{2}\left|\left(\partial - 2ie\mathbf{A}\right)\Delta\right|^2 + u|\Delta|^4\right] \\ r &= n\frac{T - T_c}{T_C}, \end{aligned}$$

if temporal fluctuations are neglected (making it a semi-classical Ginzburg-Landau theory). This brings us back to the phenomenological theory you saw in class.

Lets see how the unique experimental properties of superconductors arise from that. Below T_c, $r < 0$, so the potential $\frac{r}{2}|\Delta|^2 + u|\Delta|^4$ has a minimum at $|\Delta|^2 = \sqrt{\frac{-r}{4u}} = \Delta_0^2$. However, the phase (i.e., the Goldstone mode) is not determined by the potential, so we write $\Delta = e^{2i\theta}\Delta_0$. Putting this back in the Ginzburg-Landau action and dropping the constant terms, we have

$$S_{GL} = 2c\Delta_0^2\beta\int dx\left(\partial\theta - e\mathbf{A}\right)^2.$$

We want to find the electromagnetic response of the system. We treat the electromagnetic field as a dynamical field, so we should also add its kinetic term $S_{Maxwell} = \frac{\beta}{2}\int dx(\nabla \times \mathbf{A})^2$ (assuming $\phi = 0$, and the field is static). The total action is

$$\frac{S[A,\theta]}{\beta} = \int dx \left[2c\Delta_0^2 \left(\partial\theta - e\mathbf{A}\right)^2 + \frac{1}{2}(\nabla \times \mathbf{A})^2\right].$$

In order to get an effective action for the A we integrate over the Goldstone mode. You already saw that explicitly in class, so I will not repeat this here, but the result is that after integrating out the θ-field, the electromagnetic field acquires a mass

$$\frac{S[A]}{\beta} = \int dx \frac{1}{2}\left[\frac{\rho_0}{m}\mathbf{A}^2 + \partial_i \mathbf{A}\partial_i \mathbf{A}\right]$$

(here we used the notations used in class $\frac{\rho_0}{m} = 4c\Delta_0^2$). Deriving the equations of motion, we get $\frac{\rho_0}{m}\mathbf{A} = \nabla^2\mathbf{A}$. Taking the curl of that equation, we get the London equation $\frac{\rho_0}{m}\mathbf{B} = \nabla^2\mathbf{B}$, which was discussed in class. In particular, it was already shown that it results in the decay of the magnetic field as we go into the bulk of the superconductor.

The second effect we want to see is the zero DC resistivity. To do that, we find the current

$$\mathbf{j}(\mathbf{r}) = \frac{\delta}{\delta\mathbf{A}(\mathbf{r})}\int dx \frac{\rho_0}{2m}\mathbf{A}^2 = \frac{\rho_0}{m}\mathbf{A}.$$

Taking the time-derivative, and working in a gauge where $\phi = 0$, so $\mathbf{E} = -i\partial_\tau \mathbf{A}$, such that

$$-i\partial_\tau \mathbf{j} = \frac{\rho_0}{m}\mathbf{E}.$$

This equation says that if we have a constant DC current there is no electric field. A system with a finite DC current and zero electric field has, by definition, zero resistivity.