Concepts in Condensed Matter Physics: Exercise 4

Due date: 04/07/22

1 Little-Parks effect

Consider a superconductor which has the geometry of a ring with radius R and width d.

1. A flux ϕ penetrates the center of the ring. Write the Ginzburg-Landau theory for the ring, explain what is the condition to be in the quasi 1D limit.
2. How does T_c depend on ϕ? What is the corresponding coherence length $\xi(\phi)$? Discuss the limit of $R > \xi$ and $R < \xi$.

2 The XY – sine-Gordon duality and the BKT critical behavior

In this question you are asked to show the equivalence between the XY model and the sine-Gordon model:

$$S_{sG} = \frac{c}{2}\int d^2x \left(\nabla\theta\right)^2 - g\int d^2x \cos\theta \tag{1}$$

where θ is a non-compact real scalar field, and re-derive the RG equations near the BKT transition.

1. Expand $Z_{sG} = \int D\theta e^{-S_{sG}}$ in powers of g explicitly and show that it has the form

$$Z_{sG} = \sum_{n=0}^{\infty} \frac{(g/2)^{2n}}{(n!)^2} \prod_{j=1}^{2n} \int d^2x_j \left\langle \exp\left(i\sum_{j=1}^{2n} (-1)^j \theta(x_j) \right) \right\rangle, \tag{2}$$

   Where the brackets denote averaging with the free part $S_0 = \frac{c}{2}\int d^2x \left(\nabla\theta\right)^2$. Hint: recall that the free part is translationally invariant such that $\left\langle \left(\prod_{a=1}^{N} e^{i\theta(x_a)}\right)\left(\prod_{b=N+1}^{N+M} e^{-i\theta(x_b)}\right)\right\rangle$ is non-zero only for $N = M$.

2. Using the properties of the Gaussian average, namely $\left\langle e^{A}\right\rangle = e^{\frac{1}{2}\left\langle A^2\right\rangle}$ for A which is a linear combination of the field θ and the following identity

$$\left\langle \left(\theta(x) - \theta(x')\right)^2 \right\rangle = \frac{C(x-x')}{x} = \frac{1}{2\pi c}\log\left|\frac{x-x'}{\xi}\right|, \tag{3}$$

show that the partition function may be written as follows

$$Z_{sG} = \sum_{n=0}^{\infty} \frac{(g/2)^{2n}}{(n!)^2} \prod_{j=1}^{2n} \int d^2 x_j \left\langle \exp\left(\frac{1}{2c} \sum_{j<i}^{2n} \sigma_i \sigma_j C(x_i - x_j) \right) \right\rangle, \tag{4}$$

where σ_i denotes the sign of the vortex and ξ is a short length cutoff. This is exactly the partition function of the Coulomb gas obtained in class.

3. Repeat the derivation of the RG differential equations near the BKT transition, namely

$$\frac{dy}{d\ell} = xy, \quad \frac{dx}{d\ell}. \tag{5}$$

What are x and y in terms of c and g? Here $\ell = \log \frac{r}{\xi}$.

4. Use the above equations to determine the screening length ξ_+ on the disordered side close to the transition. Do this by estimating the value of the running parameter ℓ at which x and y reach order 1. Explain physically why ξ_+ is the screening length.

5. Obtain the superfluid stiffness J as a function of $t = T - T_c$ and show that it has a universal jump at Tc.

3 RG analysis for a localized perturbation

In this question, we will analyze the effect of local terms on a 1D superfluid. Start from a one-dimensional quantum system described by the following Ginzburg-Landau action:

$$S = \int_0^{1/T} d\tau \int dx \psi^* \left(\partial_\tau - \frac{\nabla^2}{2m} + \alpha + \beta |\psi|^2 \right) \psi. \tag{6}$$

1. Write the complex field ψ in a polar representation, $\psi = \sqrt{\rho} e^{i\theta}$. Write the corresponding action in terms of the polar representation. What is the mean-field value of ρ (denoted by ρ_0)? Show that the action has a global gauge symmetry of the form $\theta \to \theta + \text{const}$.

2. Since the above is a continuous symmetry, the θ-field is a Goldstone mode. Write $\rho = \rho_0 + \delta\rho$, expand the action to second order in $\delta\rho$ and integrate out the massive Gaussian fluctuations $\delta\rho$. Neglecting higher orders, show that the action of this Goldstone mode

then takes the form

$$S_{XY} = -\frac{1}{2}\int_0^\beta d\tau \int dx\theta \left(\chi\partial_\tau^2 + \rho_s \nabla^2\right). \quad (7)$$

This action describes the low energy excitations of a superfluid. Go to zero temperature, and rescale x, τ by defining $\tau = \tilde{\tau}/\sqrt{\chi}$, $x = \tilde{x}/\sqrt{\rho_s}$ such that the action becomes rotationally symmetric in $x - \tau$ plane.

3. Add a term of the form

$$S_1 = -u \int_0^\infty d\tilde{\tau} \cos\theta\left(\tilde{x} = 0, \tilde{\tau}\right) \quad (8)$$

to the action. Integrate out the fields at $\tilde{x} \neq 0$. Hint: you can do this for $u = 0$ and add a finite u in the end. Write an effective action for the field $\theta\left(\tilde{x} = 0, \tilde{\tau}\right)$. Show that the effective action is given by: $S_{eff} = S_0 + S_1$ with $S_0 = r \int d\omega \left|\omega\right| \theta\left(0, \omega\right) \theta\left(0, -\omega\right)$. What is the value of r? Hint: you can find that by calculating $\langle \theta\left(0, \omega\right) \theta\left(0, -\omega\right) \rangle$ in the original model and the effective model. Use the relation $\theta\left(x, \tau\right) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int dk d\omega \theta\left(k, \omega\right) e^{ikx + i\omega\tau}$.

4. Using the effective theory obtain the RG equation for u (to first order).

5. Draw the flow diagram.