Concepts in Condensed Matter Physics: Exercise 4

Due date: 14/07/24

1 The XY – sine-Gordon duality and the BKT critical behavior

In this question you are asked to show the equivalence between the XY model and the sine-Gordon model:

$$S_{SG} = \frac{c}{2}\int d^2x\,(\nabla\theta)^2 - g\int d^2x\cos\theta \tag{1}$$

where θ is a non-compact real scalar field, and re-derive the RG equations near the BKT transition.

1. Expand $Z_{SG} = \int D\theta e^{-S_{SG}}$ in powers of g explicitly and show that it has the form

$$Z_{SG} = \sum_{n=0}^{\infty}\frac{(g/2)^{2n}}{(n!)^2}\prod_{j=1}^{2n}\int d^2x_j\left\langle \exp\left(i\sum_{j=1}^{2n}(-1)^j\,\theta(x_j)\right)\right\rangle, \tag{2}$$

   Where the brackets denote averaging with the free part $S_0 = \frac{c}{2}\int d^2x\,(\nabla\theta)^2$. Hint: recall that the free part is translationally invariant such that $\left\langle\left(\prod_{a=1}^{N} e^{i\theta(x_a)}\right)\left(\prod_{b=N+1}^{N+M} e^{-i\theta(x_b)}\right)\right\rangle$ is non-zero only for $N = M$.

2. Using the properties of the Gaussian average, namely $\langle e^A\rangle = e^{\frac{1}{2}\langle A^2\rangle}$ for A which is a linear combination of the field θ and the following identity

$$\left\langle(\theta(x)-\theta(x'))^2\right\rangle = \frac{C(x-x')}{c} = \frac{1}{2\pi c}\log\left|\frac{x-x'}{\xi}\right|, \tag{3}$$

   show that the partition function may be written as follows

$$Z_{SG} = \sum_{n=0}^{\infty}\frac{(g/2)^{2n}}{(n!)^2}\prod_{j=1}^{2n}\int d^2x_j\exp\left(\frac{1}{2c}\sum_{j<i}^{2n}\sigma_i\sigma_j C(x_i-x_j)\right), \tag{4}$$

   where σ_i denotes the sign of the vortex and ξ is a short length cutoff. This is exactly the partition function of the Coulomb gas obtained in class.

3. Repeat the derivation of the RG differential equations near the BKT transition, namely

$$\frac{dy}{d\ell} = xy, \quad \frac{dx}{d\ell} = y^2. \tag{5}$$

What are x and y in terms of c and g? Here $\ell = \log \frac{r}{\xi}$.

4. Use the above equations to determine the screening length ξ_+ on the disordered side close to the transition. Do this by estimating the value of the running parameter ℓ at which x and y reach order 1. Explain physically why ξ_+ is the screening length.

5. Obtain the superfluid stiffness J as a function of $t = T - T_c$ and show that it has a universal jump at T_c.

2 Self consistent Harmonic approximation

In this question, we will do a variational approximation to estimate the critical temperature, and the scaling of the gap in the sine-Gordon (SG) model. We will show that an upper-bound of the free energy can be found by taking expectation values with respect to the gaussian action

$$S_{\text{var}} = \int d^2x \left[\frac{c}{2}(\nabla\theta)^2 + m^2\theta^2\right] \tag{6}$$

$$Z_0 = \int \mathcal{D}\theta e^{-S_{\text{var}}}, \tag{7}$$

where m is a variational parameter, and c is the same as in the SG model. We can write the exact expression

$$Z_{\text{SG}} = \int \mathcal{D}\theta e^{-S_{\text{SG}}} = \int \mathcal{D}\theta e^{-S_{\text{var}}} \left[e^{S_{\text{var}}-S_{\text{SG}}}\right] = Z_0 \langle e^{S_{\text{var}}-S_{\text{SG}}}\rangle_{\text{var}}, \tag{8}$$

and approximate it by

$$Z_{\text{SG}} \approx Z_{\text{var}} = Z_0 e^{\langle S_{\text{Var}}-S_{\text{SG}}\rangle_{\text{var}}}. \tag{9}$$

Next, we define the exact and approximated Free energies

$$F_{\mathrm{SG}} = -\frac{1}{\beta}\log\left(Z_{\mathrm{SG}}\right) \tag{10}$$

$$F_{\mathrm{var}} = -\frac{1}{\beta}\log\left(Z_{\mathrm{var}}\right) = F_0 + \frac{1}{\beta}\langle S_{\mathrm{SG}} - S_{\mathrm{Var}}\rangle_{\mathrm{var}} \tag{11}$$

1. Show that F_{var} is strictly an upper bound for F_{SG}. Can you relate this fact to the definition of the free energy and of the canonical ensemble?

2. Compute $\partial(\beta F_{\mathrm{var}})/\partial m^2$ and find m that minimizes F_{var}. (You may have to introduce an ultraviolet cutoff Λ to the calculation)

3. How can we identify the phase transition point from this calculation? hint: How does the correlator $\langle e^{i\theta(r)}e^{-i\theta(0)}\rangle$ decay for finite m? How does it decay for $m \to 0$

4. Take $\Lambda \to \infty$ and find the critical value of c. Show that $m \propto \Lambda^{\alpha}$ next to the transition, and find α. How does this compare to the results from RG analysis of the BKT transition?

3 RG analysis for a localized perturbation

In this question, we will analyze the effect of local terms on a 1D superfluid. Start from a one-dimensional quantum system described by the following Ginzburg-Landau action:

$$S = \int_0^{1/T} d\tau \int dx \psi^* \left(\partial_\tau - \frac{\nabla^2}{2m} + \alpha + \beta \left|\psi\right|^2\right)\psi. \tag{12}$$

1. Write the complex field ψ in a polar representation, $\psi = \sqrt{\rho}e^{i\theta}$. Write the corresponding action in terms of the polar representation. What is the mean-field value of ρ (denoted by ρ_0)? Show that the action has a $U(1)$ symmetry acting as $\theta \to \theta + \text{const}$.

2. Since the above is a continuous symmetry, the θ-field is a Goldstone mode. Write $\rho = \rho_0 + \delta\rho$, expand the action to second order in $\delta\rho$ and integrate out the massive Gaussian fluctuations $\delta\rho$. Neglecting higher orders, show that the action of this Goldstone mode then takes the form

$$S_{XY} = -\frac{1}{2}\int_0^{\beta} d\tau \int dx \theta \left(\chi\partial_\tau^2 + \rho_s \nabla^2\right)\theta. \tag{13}$$

This action describes the low energy excitations of a superfluid. Go to zero temperature, and rescale x, τ by defining $\tau = \tilde{\tau}/\sqrt{\chi}$, $x = \tilde{x}/\sqrt{\rho_s}$ such that the action becomes rotationally symmetric in $x - \tau$ plane.

3. Add a term of the form

$$S_1 = -u \int_0^\infty d\tilde{\tau} \cos\theta\left(\tilde{x} = 0, \tilde{\tau}\right) \tag{14}$$

to the action. Integrate out the fields at $\tilde{x} \neq 0$. Hint: you can do this for $u = 0$ and add a finite u in the end. Write an effective action for the field $\theta\left(\tilde{x} = 0, \tilde{\tau}\right)$. Show that the effective action is given by: $S_{eff} = S_0 + S_1$ with $S_0 = r \int d\omega \left|\omega\right| \theta\left(0, \omega\right) \theta\left(0, -\omega\right)$. What is the value of r? Hint: you can find that by calculating $\left\langle \theta\left(0, \omega\right) \theta\left(0, -\omega\right) \right\rangle$ in the original model and the effective model. Use the relation $\theta\left(x, \tau\right) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int dk d\omega \theta\left(k, \omega\right) e^{ikx + i\omega\tau}$.

4. Using the effective theory obtain the RG equation for u (to first order).

5. Draw the flow diagram.