Block bases of the Haar system as complemented subspaces of L_p, $2 < p < \infty$ *

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Abstract

It is shown that the span of $\{a_i h_i \oplus b_i e_i\}_{i=1}^n$, where $\{h_i\}$ is the Haar system in L_p and $\{e_i\}$ the canonical basis of ℓ_p, is well isomorphic to a well complemented subspace of L_p, $2 < p < \infty$. As a consequence we get that there is a rearrangement of the (initial segments of the) Haar system in L_p, $2 < p < \infty$, any block basis of which is well isomorphic to a well complemented subspace of L_p.

1 Introduction

Recall that for the dyadic interval $I = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ the Haar function h_I is defined to be

$$h_I(t) = \begin{cases} +1, & \text{if } t \text{ is in the left half of } I, \\ -1, & \text{if } t \text{ is in the right half of } I. \end{cases}$$

The usual order of the Haar system is the lexicographic order on $\{(n,i)\}$. The main motivation of the present note comes from [MS] in which another useful order is defined: $I \preceq J$ if either I and J are disjoint and I is to left of J, or I is contained in J. This order is more correlated with the order on the interval $[0,1]$ than the lexicographic order and as such is also natural. Its drawback is that unlike the natural order it is not a well ordering on the infinite Haar system. In [MS] it is proved that any block basis of (a finite piece of) the Haar system in L_p, $2 < p < \infty$, in this new order is equivalent, with constant depending only on p, to a sequence of the form $\{a_i h_i \oplus b_i e_i\}$ for some scalars $\{a_i, b_i\}$ and some subsequence $\{h_i\}$ of the original Haar system. Here e_i denotes the unit vector basis of ℓ_p and $\oplus$ an ℓ_p sum. This was used in [MS] to solve a problem of [DS] by showing that there is an unconditional basic sequence in L_p not equivalent to the ℓ_p basis yet not containing block bases uniformly equivalent to the unit vector basis of ℓ_2^n.

The purpose of the present note is to prove that any sequence of the form $\{a_i h_i \oplus b_i e_i\}_{i=1}^k$ spans in L_p, $2 < p < \infty$, a space well isomorphic to a well complemented subspace of L_p. As an immediate consequence one gets that any block basis of (a finite portion of) $\{h_I\}$ with the order $\preceq$ is well isomorphic to a well complemented subspace of L_p. The question of whether the span of any finite sequence of the form $\{a_i h_i \oplus b_i e_i\}_{i=1}^k$ is well isomorphic to ℓ_p^k is left open.

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2 Preliminaries

In this section we gather a few known results that will be used in the sequel. We only present a proof of one of them (Theorem 2.2) which was not well circulated before. The first theorem, due to H.P. Rosenthal together with its proof (involving an inequality for p-th moments of sums of independent random variables) proved to be an extremely useful result.

Theorem 2.1 *([R]) Let $2 < p < \infty$, let $\{f_i\}_{i=1}^{\infty}$ be a sequence of independent symmetric three valued random variables, and let Y_p denote their closed linear span in L_p. Then the orthogonal projection P from L_p onto Y_p is bounded by a constant K_p depending only on p.*

The next theorem appears only in [Sc], we thus include its simple proof.

Theorem 2.2 *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of measurable subsets of $[0,1]$ such that for every $i \neq j$, $A_i \cap A_j = \phi$ or $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Then the span of $\{r_i \otimes \chi_{A_i}\}_{i=1}^{\infty}$ is complemented in $L_p([0,1]^2)$, $1 < p < \infty$ by means of the orthogonal projection. Moreover the norm of the projection depends only on p. Here r_i is the i-th Rademacher function and $f \otimes g(s,t) = f(s)g(t)$.*

Proof: We shall need a result of E. Stein [St]: *For every $1 < p < \infty$ there is a constant A_p, such that for every increasing sequence of σ-fields in $[0,1]$, $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$, and for every sequence $\{f_k\}_{k=1}^{\infty}$ of functions in L_p, we have:*

$$\Big\| \Big(\sum_{k=1}^{\infty} |E_k(f_k)|^2 \Big)^{1/2} \Big\|_p \le A_p \Big\| \Big(\sum_{k=1}^{\infty} |f_k|^2 \Big)^{1/2} \Big\|_p , \tag{1}$$

where $E_k(f)$ is the conditional expectation of f with respect to $\mathcal{F}_k$.

The proof of Stein's result is simple so we sketch it as well: By Doob's inequality,

$$\| \sup |E_k(f_k)| \|_p \le \| \sup_k E_k(\sup_l |f_l|) \|_p \le A_p \| \sup_l |f_l| \|_p .$$

Clearly also,

$$\Big\| \Big(\sum_{k=1}^{\infty} |E_k(f_k)|^p \Big)^{1/p} \Big\|_p \le \Big\| \Big(\sum_{k=1}^{\infty} |f_k|^p \Big)^{1/p} \Big\|_p$$

and thus, by interpolation, we get (1) for $1 < p < 2$. The case $2 < p < \infty$ follows by duality.

To prove Theorem 2.2 it is enough to show that for every n and $i_1, \dots, i_n$ there is a projection from $L_p \otimes L_p = L_p([0,1]^2)$ onto $[r_{i_j} \otimes \chi_{A_{i_j}}]_{j=1}^n$ with norm that doesn't depend on n and $i_1, \dots, i_n$. Since for each $i_1, \dots, i_n$ there is a projection from $L_p \otimes L_p$ onto $[r_{i_j}]_{j=1}^n \otimes L_p$ with norm that doesn't depend on $i_1, \dots, i_n$, it is enough to show existence of such a projection from $[r_{i_j}]_{j=1}^n \otimes L_p$ onto $[r_{i_j} \otimes \chi_{A_{i_j}}]_{j=1}^n$. Given $i_1, \dots, i_k$ we may assume without loss of generality that if $1 \le k < \ell \le n$ then $A_{i_k} \cap A_{i_\ell} = \phi$ or $A_{i_k} \supseteq A_{i_\ell}$. Let $\mathcal{F}_k$ be the σ-field generated by $A_{i_1}, \dots, A_{i_k}$, $1 \le k \le n$. Then $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_n$ and from the assumption above it is clear that A_{i_k} is an atom of $\mathcal{F}_k$ for every $1 \le k \le n$.

We define:

$$P : [r_{i_j}]_{j=1}^n \otimes L_p \overset{\text{onto}}{\longrightarrow} [r_{i_j} \otimes \chi_{A_{i_j}}]_{j=1}^n$$

by

$$P\big(\sum_{j=1}^{n} r_{i_j}\otimes f_j\big)=\sum_{j=1}^{n} r_{i_j}\otimes (E_{i_j}f_i)_{|A_{i_j}}\ .$$

From the above it is clear that P is well defined and that it is a projection.

We are left to prove that $\|P\|\le K_p$, where K_p depend only on p. From Khintchine's inequality we may conclude the existence of a constant B_p which depends only on p, such that:

$$\big\|\sum_{j=1}^{n} r_{i_j}\otimes (E_{i_j}f_j)_{|A_{i_j}}\big\|_p \overset{B_p}{\approx} \big\|\big(\sum_{j=1}^{n}|(E_{i_j}f_j)_{|A_{i_j}}|^2\big)^{1/2}\big\|_p$$

and

$$\big\|\sum_{j=1}^{n} r_{i_j}\otimes f_j\big\|_p \overset{B_p}{\approx} \big\|\big(\sum_{j=1}^{n}|f_j|^2\big)^{1/2}\big\|_p$$

while

$$\big\|\big(\sum_{j=1}^{n}|(E_{i_j}f_j)_{|A_{i_j}}|^2\big)^{1/2}\big\|_p\le \big\|\big(\sum_{j=1}^{n}|(E_{i_j}f_j)|^2\big)^{1/2}\big\|_p\ .$$

From this and from Stein's theorem the existence of the constant K_p follows. ■

The last result we state here is a theorem of Burkholder which in turn generalizes the main inequality of [R] from the setting of independent random variables to that of martingale differences.

Theorem 2.3 *([B]) Let $2<p<\infty$ and let $\{f_i\}_{i=1}^{\infty}$ be a martingale with respect to the increasing sequence of σ-fields $\{\mathcal{E}_i\}_{i=1}^{\infty}$. Then for $d_n=f_{n+1}-f_n$, the martingale difference, we have*

$$\big\|\sum_{i=1}^{\infty} a_i d_i\big\|_p \overset{C_p}{\approx} \max\big\{\big\|\big(\sum_{i=1}^{\infty}|a_i|^2E(d_i^2|\mathcal{E}_i)\big)^{\frac12}\big\|_p,\big(\sum_{i=1}^{\infty}|a_i|^p\|d_i\|_p^p\big)^{\frac1p}\big\}\ ,$$

where C_p is a constant depending only on p.

3 The main result

The main technical result here is the following theorem

Theorem 3.1 *Let $\{h_i\}_{i=1}^{n}$ be a subsequence of the Haar system $\{h_I\}_{I\in T_N}$ (ordered in its natural order). Let $\{g_i\}_{i=1}^{n}$ be a sequence of functions on $[0,1]$ with the following properties:*

1. *g_i is symmetric three valued random variable on $[0,1]$ for all $i=1,\dots,n$.*
2. $(\mathrm{supp} g_k)\cap\big(\frac{j-1}{2^N},\frac{j}{2^N}\big)\ne\phi \Leftrightarrow (\mathrm{supp} h_k)\cap\big(\frac{j-1}{2^N},\frac{j}{2^N}\big)\ne\phi$, $j=1,\dots,2^N$, $k=1,\dots,n$.
3. *If $(\mathrm{supp} g_k)\cap\big(\frac{i-1}{2^N},\frac{i}{2^N}\big)\ne\phi$ and $(\mathrm{supp} g_k)\cap\big(\frac{j-1}{2^N},\frac{j}{2^N}\big)\ne\phi$, $i,j=1,\dots,2^N$, $k=1,\dots,n$, then*
$$g_{k|_{\left(\frac{i-1}{2^N},\frac{i}{2^N}\right)}}(x)=g_{k|_{\left(\frac{j-1}{2^N},\frac{j}{2^N}\right)}}\big(x-\frac{i-j}{2^N}\big)\ .$$

4. *For each* $j = 1, \dots, 2^N$, $\left\{ g_{i|\left(\frac{j-1}{2^N}, \frac{j}{2^N}\right)} \right\}_{i=1}^{n}$ *are independent as random variable on the probability space* $\left(\frac{j-1}{2^N}, \frac{j}{2^N}\right)$ *with normalized Lebesgue measure.*

Then the span of $\{g_i\}_{i=1}^n$ *is well complemented in* L_p, $p > 2$, *i.e., there is a projection* P *from* L_p *onto span* $\{g_i\}_{i=1}^n$ *whose norm depends only on* p.

Proof: Assume, as we may that non of the g_i-s is the zero function. For each i let $j(i)$ be such that $(\text{supp} g_i) \cap \left[\frac{j(i)-1}{2^N}, \frac{j(i)}{2^N}\right] \neq \phi$. We look at the σ-field generated by the sets $\left\{(\text{supp} g_i) - \frac{j(i)-1}{2^N}\right\}_{i=1}^n$ and $\left[0, \frac{1}{2^N}\right]$, and suppose that the atoms inside $\left[0, \frac{1}{2^N}\right]$ are $\{A_j\}_{j=1}^m$ (m is a finite positive number). We define $\tilde{\chi}_{A_j} = \sum_{k=0}^{2^N-1} \chi_{A_j}(x - \frac{1}{2^N}k)$ and $g_{i,j} = g_i \cdot \tilde{\chi}_{A_j}$ (some of the $g_{i,j}$, $j = 1, \dots, m$, can be the zero function, but this will not effect the argument bellow). Then $\sum_{j=1}^m g_{i,j} = g_i$, $i = 1, \dots, n$. Define $V = span\{g_i | i = 1, \dots, n\}$ and $V_1 = span\{g_{i,j} | i = 1, \dots, n \ j = 1, \dots, m\}$ so that we have $V_1 \supseteq V$. By property 2 in the statement of the theorem we have that the system $\{g_{i,j}\}_{i=1}^{n}{}_{j=1}^{m}$ satisfies the conditions of theorem 2.2 (Note that $\{g_{i,j}\}_{i=1}^{n}{}_{j=1}^{m}$ has the same distribution as $\{r_{i,j} \otimes |g_{i,j}|\}_{i=1}^{n}{}_{j=1}^{m}$), so we may conclude that $\{g_{i,j}\}_{i=1}^{n}{}_{j=1}^{m}$ is well complemented, which means that there is a projection P_1 from $L_p([0,1])$ onto V_1 with norm depending only on p.

It is enough to show that we can find a good projection P_2 from V_1 onto V. Recall that for each j the system $\left\{ g_{i|\left(\frac{j-1}{2^N}, \frac{j}{2^N}\right)} \right\}_{i=1}^{n}$ is composed of independent three valued symmetric random variables (on the probability space $(\frac{j-1}{2^N}, \frac{j}{2^N})$). So by applying the orthogonal projection $P_{2,j}$ on each interval of the form $(\frac{j-1}{2^N}, \frac{j}{2^N})$, $j = 1, \dots, 2^N$, we get, using Rosenthal's theorem 2.1, a bounded operator with norm depending only on p. We then define $P_2 = \sum_{j=1}^{2^N} P_{2,j}$ and it is easy to check that P_2 is also bounded with the same bound.

The only thing still to check is that P_2 is indeed into V. That this is indeed the case follows from the orthogonality of the $P_{2,j}$-s. More precisely this follows from the following two facts:

1. Each $g_{i,j}$ is a duplicate of one function (property 3).

2. When we project $g_{i,j}$ onto V the only component that will not be sent to zero is that of g_i (this follows immediately from the fact that

$$\langle g_{i,j}, g_k \rangle = \int g_{i,j} \cdot g_k \, dx \neq 0 \text{ iff } i = k) \ .$$

This concludes the proof of Theorem 3.1. ■

Remark: Inspecting the proof, it is easy to see that the Haar system was used here only superficially, actually only the supports of the Haar functions play a role here. It is also easy to see that these supports could be replaces by any sequence of subsets of $[0,1]$, $\{A_n\}_{n=1}^\infty$, satisfying for every $i \neq j$, $A_i \cap A_j = \phi$ or $A_i \subseteq A_j$ or $A_j \subseteq A_i$.

Next we would like to show that any sequence of the form $\{a_i h_i \oplus b_i e_i\}_{i=1}^n$ is well equivalent in L_p, $p > 2$, to a sequence with the properties of the assumptions of Theorem 3.1. For this we need the following proposition.

Proposition 3.2 *Let $\{h_i\}_{i=1}^n$ be a subsequence of the Haar system $\{h_I\}_{I\in T_N}$ (ordered in its natural order). For every $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ sequences of positive numbers, we can find a sequence of functions $\{g_i\}_{i=1}^n$ with the properties of Theorem 3.1 and with the following additional property*

$$E(g_i^2|\mathcal{E}_i) = a_i^2 h_i^2 \quad , \quad \|g_i\|_p^p = a_i^p |I_i| + b_i^p \ ,$$

where $I_i = \mathrm{supp} h_i$, and $\mathcal{E}_i$ is the σ-field generated by $\{g_1, \dots, g_{i-1}\}$ and $\left\{[\frac{j-1}{2^N}, \frac{j}{2^N}]\right\}_{j=1}^{2^N}$. Here $\frac{1}{2^N}$ is the size of the smallest of $|I_i| = |\mathrm{supp} h_i|$, $i = 1, \dots, n$.

The proof of the Proposition follows from the next lemma applied to each interval of the form $(\frac{i-1}{2^N}, \frac{i}{2^N})$, $i = 1, \dots, 2^N$ inside $\mathrm{supp} h_i$.

Lemma 3.3 *Let a, b be positive numbers, and $I \in \left\{\frac{1}{2^n};\ n = 0, 1, \dots, N\right\}$. We can find $c > 0$ and $0 < d \le \frac{1}{2^N}$ such that the function $f = c\chi_{[0,d]}$ satisfies $E(f^2|[0, \frac{1}{2^N}]) = a^2 \chi_{[0, \frac{1}{2^N}]}$ and $\|f\|_p^p = \frac{a^p I + b^p}{2^N I}$.*

Proof: We need to solve the following two equations

$$\frac{c^2 d}{1/2^N} = a^2 \ , \ c^p d = \frac{a^p I + b^p}{2^N I} \ .$$

The solution is given by

$$c = \left(\frac{a^p I + b^p}{a^2 I}\right)^{\frac{1}{p-2}} \ , \ d = \left(\frac{a^p I}{a^p I + b^p}\right)^{\frac{2}{p-2}} \cdot \frac{1}{2^N} \ ,$$

and indeed $d \le 1/2^N$. ■

We are now ready to state and prove the main theorem

Theorem 3.4 *Let $2 < p < \infty$. There exists a constant $0 < K_p < \infty$ such that if $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ are two sequences of numbers and $\{h_i\}_{i=1}^n$ is a subsequence of the Haar system (in its natural order) then $\{a_i h_i \oplus b_i e_i\}_{i=1}^n$ spans a space isomorphic, with constant at most K_p, to a K_p complemented subspace of L_p. ($\{e_i\}$ is the canonical ℓ_p basis).*

Proof: Using Proposition 3.2 we build the sequence $\{g_i\}_{i=1}^n$. We then have, (using Burkholder's theorem 2.3, say, although this can be easily avoided here)

$$\begin{aligned} \Big\|\sum_{i=1}^n \alpha_i (a_i h_i \oplus b_i e_i)\Big\|_p^p &= \Big\|\sum_{i=1}^n \alpha_i a_i h_i\Big\|_p^p + \Big\|\sum_{i=1}^n \alpha_i b_i e_i\Big\|_p^p \\ &= \Big\|\sum_{i=1}^n \alpha_i a_i h_i\Big\|_p^p + \sum_{i=1}^n |\alpha_i|^p |b_i|^p \\ &\overset{C_p}{\approx} \Big\|\Big(\sum_{i=1}^n |\alpha_i|^2 |a_i|^2 h_i^2\Big)^{1/2}\Big\|_p^p + \sum_{i=1}^n |\alpha_i|^p |a_i|^p |I_i| + \sum_{i=1}^n |\alpha_i|^p |b_i|^p \ . \end{aligned}$$

Using Burkholder's theorem 2.3 again,

$$\Big\|\sum_{i=1}^{n}\alpha_i g_i\Big\|_p^p \overset{C_p}{\approx} \Big\|\Big(\sum_{i=1}^{n}|\alpha_i|^2 E(g_i^2|\mathcal{E}_i)\Big)^{1/2}\Big\|_p^p+\sum_{i=1}^{n}|\alpha_i|^p\|g_i\|_p^p .$$

By Proposition 3.2, $E(g_i^2|\mathcal{E}_i)=|a_i|^2h_i^2$ and $\|g_i\|_p^p=|a_i|^p|I_i|+|b_i|^p$ and thus $\{g_i\}_{i=1}^n$ is equivalent, with constant depending only on p, to $\{a_ih_i\oplus b_ie_i\}_{i=1}^n$. Finally, by Theorem 3.1, the span of $\{g_i\}_{i=1}^n$ is well complemented in L_p. ∎

Remark: Inspecting the proof of this theorem, and using the remark following the proof of Theorem 3.1 it is easy to see that the Haar system could be replaced with any sequence of the form $\{r_i\otimes\chi_{A_i}\}$ where $\{A_i\}_{i=1}^\infty$ is a sequence of subsets of $[0,1]$ satisfying for every $i\neq j$, $A_i\cap A_j=\phi$ or $A_i\subseteq A_j$ or $A_j\subseteq A_i$. We do not dwell on it here because a more general fact holds as is explained in the remark concluding this note.

Using this theorem and the main result of [MS] we get

Corollary 3.5 *For every $p>2$ there exists a constant $K_p<\infty$, such that for all N, every block basis of $\{h_I\}_{I\in T_N}$ (with respect to the order $\preceq$) spans a space isomorphic, with constant at most K_p, to a complemented subspace of L_p with projection of norm at most K_p.*

There are two results proved after the distribution of a preliminary version of this work.

1. Paul Müller extended the result here to the setting of the space VMO. Our case of L_p follows from this as well by interpolation, thus providing another proof of the main result here. He also characterized the infinite dimensional spaces obtained: Each space spanned by a sequence of the form $\{a_ih_i\oplus b_ie_i\}_{i=1}^\infty$ is isomorphic to either L_p or ℓ_p. The finite version of this characterization is still open.

2. W.B. Johnson and the second named author observed that it follows from the results here (and the method of proof of [Sc2]) that if $\{x_i\}$ is (finite or infinite) unconditional basic sequence in L_p, $2<p<\infty$, spanning a well complemented subspace, then, for any set of scalars $\{a_i\}$, $\{x_i\oplus a_ie_i\}$ spans a space well isomorphic to a well complemented subspace of L_p. The constants of isomorphism and complementation depend only on p, the norm of the projection onto span$\{x_i\}$ and the unconditionality constant of $\{x_i\}$.

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