An example related to the slicing inequality for general measures

Bo'az Klartag and Alexander Koldobsky

Abstract

For $n \in \mathbb{N}$, let S_n be the smallest number $S > 0$ satisfying the inequality

$$\int_K f \leq S \cdot |K|^{\frac{1}{n}} \cdot \max_{\xi \in S^{n-1}} \int_{K \cap \xi^\perp} f$$

for all centrally-symmetric convex bodies K in $\mathbb{R}^n$ and all even, continuous probability densities f on K. Here $|K|$ is the volume of K. It was proved in [16] that $S_n \leq 2\sqrt{n}$, and in analogy with Bourgain's slicing problem, it was asked whether S_n is bounded from above by a universal constant. In this note we construct an example showing that $S_n \geq c\sqrt{n}/\sqrt{\log \log n}$, where $c > 0$ is an absolute constant. Additionally, for any $0 < \alpha < 2$ we describe a related example that satisfies the so-called ψ_α-condition.

1 Introduction

Suppose that $K \subseteq \mathbb{R}^n$ is a centrally-symmetric convex set of volume one (i.e., $K = -K$). Given an arbitrary continuous probability density $f : K \to \mathbb{R}$, can we find a hyperplane $H \subseteq \mathbb{R}^n$ passing through the origin such that

$$\int_{H \cap K} f \geq c$$

where $c > 0$ is a universal constant, which is in particular independent of K, f and even the dimension n?

For many classes of convex bodies, the answer is surprisingly positive. It was proven by the second-named author [17] that the answer is affirmative in the case where $K \subseteq \mathbb{R}^n$ is *unconditional*, i.e.,

$$(x_1, \ldots, x_n) \in K \quad \iff \quad (|x_1|, \ldots, |x_n|) \in K \qquad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

This generalizes a result first proven by Bourgain [3], who considered the case where the density f is constant. Bourgain's investigations have led to the formulation of the slicing problem [3, 4], which asks whether $\sup_n L_n < \infty$, where $L_n > 0$ is the minimal number L such that for any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$,

$$|K|_n \leq L \cdot \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|_{n-1} \cdot |K|_n^{1/n}.$$

Here $\xi^\perp$ is the central hyperplane perpendicular to the vector $\xi \in S^{n-1}$, and $S^{n-1} = \{x \in \mathbb{R}^n\,;\, |x| = 1\}$ is the Euclidean unit sphere centered at the origin. We write $|K|_n$ for the n-dimensional volume of K. When the dimension is clear from the context, we will simply use $|K|$ in place of $|K|_n$. Bourgain's slicing problem is still unsolved, the best-to-date estimate $L_n \leq Cn^{1/4}$ was established by the first-named author [11], removing a logarithmic term from an earlier estimate by Bourgain [5]. In analogy with the slicing problem, for $n \geq 1$ let S_n be the smallest number $S > 0$ satisfying the inequality

$$\mu(K) \leq S \cdot \max_{\xi \in S^{n-1}} \mu^+(K \cap \xi^\perp) \cdot |K|^{\frac{1}{n}} \tag{1}$$

for all centrally-symmetric convex bodies $K \subseteq \mathbb{R}^n$, and all measures μ with a non-negative continuous density f in $\mathbb{R}^n$. Here we abbreviate

$$\mu^+(K \cap \xi^\perp) = \int_{K \cap \xi^\perp} f$$

where the restriction of the density f to $\xi^\perp$ is integrated with respect to the Lebesgue measure in $\xi^\perp$.

Many of the positive results towards the slicing problem may be generalized from the case of the uniform measure on a convex domain K to the broader class of any continuous probability density on K. Thus (1) holds true, with S having the order of magnitude of a universal constant, whenever K is the polar to a convex body with bounded volume ratio (see [17]) or the unit ball of a subspace of L_p with $p > 2$ (see [18]). The first result of this kind was proved in [15]: If K belongs to the class of intersection bodies $\mathcal{I}_n$ (see definition in Section 2), then (1) holds with $S = 2$ for all measures with even continuous densities.

In view of the positive results mentioned above, one could think that perhaps $\sup_n S_n < \infty$. In this note we show that this is not the case, and prove the following:

Theorem 1.1. *There exist universal constants $c, C > 0$ so that for any $n \geq 3$,*

$$\frac{c\sqrt{n}}{\sqrt{\log \log n}} \leq S_n \leq C\sqrt{n}.$$

The new result here is the left-hand side estimate. The right-hand side estimate was first established in [16], and later a different proof was found in [6] where the central-symmetry assumption was no longer required. In fact, the upper estimate for the constants S_n may be deduced from the following theorem proved in [17, Corollary 1], which we now describe.

A compact $K \subseteq \mathbb{R}^n$ is star-shaped if $tK \subseteq K$ for $0 \leq t \leq 1$, where $tK = \{tx\,;\, x \in K\}$. We say that a star-shaped K is a star body if its radial function

$$\rho_K(x) = \max\{a \geq 0 :\ ax \in K\} \qquad\qquad (x \in S^{n-1})$$

is continuous and positive in S^{n-1}. For a star body $K \subseteq \mathbb{R}^n$ denote by

$$d_{\mathrm{ovr}}(K, \mathcal{I}_n) = \inf\left\{ \left(\frac{|D|}{|K|}\right)^{1/n} ;\ K \subseteq D,\ D \in \mathcal{I}_n \right\}$$

the outer volume ratio distance from K to the class of intersection bodies $\mathcal{I}_n$.

Theorem 1.2. *For any $n \in \mathbb{N}$, any centrally-symmetric star body $K \subseteq \mathbb{R}^n$, and any measure μ with a continuous density on K,*

$$\mu(K) \leq 2\ d_{\mathrm{ovr}}(K, \mathcal{I}_n) \cdot \max_{\xi \in S^{n-1}} \mu^+(K \cap \xi^\perp) \cdot |K|^{1/n}.$$

The right-hand estimate of Theorem 1.1 follows from Theorem 1.2 and John's theorem, since all elllipsoids are intersection bodies (see [16]). For the sake of completeness, we present a short proof of Theorem 1.2 and related results in Section 2. In Section 3 we move on to discuss the lower estimate for S_n which shows that the $\sqrt{n}$ upper bound is in fact optimal up to a $\log\log$-term:

Theorem 1.3. *For any $n \geq 3$ there exists a centrally-symmetric convex body $T \subseteq \mathbb{R}^n$ and an even, continuous probability density $f : T \to [0, \infty)$ such that for any affine hyperplane $H \subseteq \mathbb{R}^n$,*

$$\int_{T \cap H} f \leq C \frac{\sqrt{\log\log n}}{\sqrt{n}} \cdot |T|^{-1/n}, \tag{2}$$

where $C > 0$ is a universal constant.

Note that the hyperplane H in Theorem 1.3 is not required to pass through the origin. The combination of Theorem 1.2 and Theorem 1.3 implies the following:

Corollary 1.4. *There exists a centrally-symmetric convex body $T \subseteq \mathbb{R}^n$ with $d_{\mathrm{ovr}}(T, \mathcal{I}_n) \geq c\sqrt{n}/\sqrt{\log\log n}$, where $c > 0$ is a universal constant.*

For $\alpha \in (0, 2]$ we say that a measure μ on $\mathbb{R}^n$ admits ψ_α-tails with parameters (β, γ) if for any linear functional $\ell : \mathbb{R}^n \to \mathbb{R}$

$$\mu\left(\{x \in \mathbb{R}^n\,;\, |\ell(x)| \geq tE\}\right) \leq \beta \exp(-\gamma t^\alpha) \cdot \mu(\mathbb{R}^n) \qquad\qquad (\text{for all } t > 0) \tag{3}$$

where $E = \int_{\mathbb{R}^n} |\ell(x)| d\mu(x)$. It follows from the Brunn-Minkowski inequality that the uniform probability measure μ on a convex body in $\mathbb{R}^n$ has ψ_1-tails with parameters (β, γ) that are universal constants, see, e.g., [2, Section 2.4]. It follows from the argument by

Bourgain (see e.g. [2, Section 3.3]) that for any measure μ with an even, continuous density supported on a centrally-symmetric convex body $K \subseteq \mathbb{R}^n$,

$$\mu(K) \leq C(\beta, \gamma) \cdot n^{(2-\alpha)/4} \log n \cdot \sup_{H \subseteq \mathbb{R}^n} \mu^+(K \cap H) \cdot |K|^{1/n}, \tag{4}$$

where the supremum runs over all $(n-1)$-dimensional affine hyperplanes $H \subseteq \mathbb{R}^n$, where μ is assumed to have ψ_α-tails with parameters (β, γ), and where $C(\beta, \gamma) > 0$ depends only on β and γ. For completeness, we provide a short argument explaining (4) in an appendix.

Specializing (4) to the case $\alpha = 1$, we obtain the bound $L_n \leq Cn^{1/4} \log n$ for Bourgain's slicing problem, which is not far from the best estimate known to date. In the log-concave case it was proven in [12] that the logarithmic factor in (4) is not needed. The following theorem establishes the near-optimality of the bound (4), up to logarithmic terms:

Theorem 1.5. *For any n and $0 < \alpha \leq 2$ there exists a centrally-symmetric convex body $T \subseteq \mathbb{R}^n$ and an even, continuous probability density $f : T \to [0, \infty)$ with the following properties:*

(i) $\displaystyle\int_{T \cap H} f \leq C_\alpha \cdot n^{(\alpha-2)/4} \cdot |T|^{-1/n}$ *for any affine hyperplane $H \subseteq \mathbb{R}^n$.*

(ii) The measure whose density is f admits ψ_α-tails with parameters $(\tilde{c}_\alpha, \tilde{C}_\alpha)$.

Here, $\tilde{c}_\alpha, C_\alpha, \tilde{C}_\alpha > 0$ depend solely on $\alpha \in (0, 2]$.

Theorem 1.5 shows that Bourgain's slicing problem cannot be resolved on the affirmative if all that is used is the uniformly subexponential tails of linear functionals on convex bodies.

Throughout this paper, unless specified otherwise we write $c, C, \tilde{C}$ etc. for various positive, universal constants, whose value is not necessarily the same in different appearances. We use lower-case $c, \tilde{c}, c_1$ for sufficiently small positive universal constants, while $C, C_1, \tilde{C}_1$ etc. are sufficiently large universal constants. A convex body K in $\mathbb{R}^n$ is a compact, convex set with a non-empty interior. The standard scalar product between $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$ or by $\langle x, y \rangle$. We write log for the natural logarithm.

Acknowledgements. We would like to thank Sergey Bobkov for encouraging us to discuss and think on this problem. We thank the anonymous referee for a thoughtful report. A major part of the work was done when both authors were visiting the Banff International Research Station during May 21–26, 2017. We would like to thank BIRS for hospitality. The first-named author was supported in part by a European Research Council (ERC) grant. The second-named author was supported in part by the US National Science Foundation grant DMS-1700036.

2 The outer volume ratio distance from intersection bodies

The *Minkowski functional* of a star body $D \subseteq \mathbb{R}^n$ is defined by

$$\|x\|_D = \min\{a \geq 0 : \ x \in aK\} \qquad\qquad (x \in S^{n-1}).$$

Note that $\|x\|_D^{-1} = \rho_D(x)$ for any $x \in S^{n-1}$, where ρ_D is the radial function of D. The class of intersection bodies was introduced by Lutwak [21]. Let D, L be origin-symmetric star bodies in $\mathbb{R}^n$. We say that D is the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the section of L by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_D(\xi) = |L \cap \xi^\perp| = \frac{1}{n-1}\int_{S^{n-1}\cap\xi^\perp} \rho_L^{n-1} d\theta = \frac{1}{n-1} R\left(\rho_L^{n-1}\right)(\xi),$$

where $R : C(S^{n-1}) \to C(S^{n-1})$ is the *spherical Radon transform*

$$Rg(\xi) = \int_{S^{n-1}\cap\xi^\perp} g(x)dx, \qquad \forall g \in C(S^{n-1}).$$

All star bodies K that appear as intersection bodies of star bodies form *the class of intersection bodies of star bodies.*

A more general class of *intersection bodies* is defined as follows; see [9]. If ν is a finite Borel measure on S^{n-1}, then the spherical Radon transform $R\nu$ of ν is a functional on $C(S^{n-1})$ acting by

$$(R\nu, g) = (\nu, Rg) = \int_{S^{n-1}} Rg(x)d\nu(x), \qquad \forall g \in C(S^{n-1}).$$

Definition 2.1. *A star body D in $\mathbb{R}^n$ is called an intersection body, and we write $D \in \mathcal{I}_n$, if there exists a finite Borel measure ν_D on S^{n-1} such that $\rho_D = R\nu_D$ as functionals on $C(S^{n-1})$, i.e.*

$$\int_{S^{n-1}} \rho_D(x)g(x)dx = \int_{S^{n-1}} Rg(x)d\nu_D(x), \qquad \forall g \in C(S^{n-1}). \tag{5}$$

For example, let us consider the cross-polytope

$$B_1^n = \left\{x \in \mathbb{R}^n \, ; \, \|x\|_1 = \sum_{k=1}^n |x_k| \leq 1\right\}.$$

It was proved in [13] that B_1^n is an intersection body. To see this, note that the function $e^{-\|\cdot\|_1}$ is the Fourier transform of the function

$$\phi(\xi) = \frac{1}{\pi^n}\prod_{k=1}^n \frac{1}{1+\xi_k^2}, \qquad\qquad (\xi \in \mathbb{R}^n),$$

and use the connection between the Radon and Fourier transforms: For $x \in S^{n-1}$,

$$\begin{aligned}\rho_{B_1^n}(x) = \|x\|_1^{-1} = \frac{1}{2}\int_{\mathbb{R}} e^{-t\|x\|_1} dt = \frac{1}{\pi^{n-1}} \int_{x^\perp} \prod_{k=1}^n \frac{1}{1+\xi_k^2} d\xi \\ = \frac{1}{\pi^{n-1}} \int_{S^{n-1}\cap x^\perp} \left(\int_0^\infty t^{n-2} \prod_{k=1}^n \frac{1}{1+t^2\xi_k^2} dt \right) d\xi.\end{aligned}$$

We get that the radial function of the cross-polytope is the spherical Radon transform of the function

$$\xi \to \frac{1}{\pi^{n-1}} \int_0^\infty t^{n-2} \prod_{k=1}^n \frac{1}{1+t^2\xi_k^2} dt.$$

This function is integrable on the sphere, but it is not bounded (it takes infinite values on a set of measure zero). Therefore, B_1^n is an intersection body, but not the intersection body of a star body; see [13] or [14, Section 4.3] for details. Note that it was proved in [13] that all polar projection bodies are intersection bodies.

It was proven in [17] that $d_{\rm ovr}(K, \mathcal{I}_n) \le e$ for every unconditional convex body K in $\mathbb{R}^n$. In fact, by a result of Lozanovskii [20] (see the proof in [22, Corollary 3.4]), there exists a linear operator T on $\mathbb{R}^n$ so that $T(B_\infty^n) \subset K \subset nT(B_1^n)$, where B_∞^n is the cube with sidelength 2 in $\mathbb{R}^n$. Let $D = nT(B_1^n)$. From the fact that a linear transformation of an intersection body is an intersection body, the body D is an intersection body in $\mathbb{R}^n$. Since $|B_1^n| = 2^n/n!$, we have $|D|^{1/n} \le 2e|\det T|^{1/n}$. On the other hand, $|T(B_\infty^n)| = 2^n|\det T|$, and $T(B_\infty^n) \subset K$, so $|D|^{1/n} \le e\ |K|^{1/n}$.

We now present a proof of Theorem 1.2 that is slightly shorter than that in [17].

Proof of Theorem 1.2. For every $\xi \in S^{n-1}$, we have

$$\mu^+(K \cap \xi^\perp) \le \max_{\theta \in S^{n-1}} \mu^+(K \cap \theta^\perp). \tag{6}$$

Let f be the continuous density of the measure μ. Writing the integral in spherical coordinates, we see that for every $\xi \in S^{n-1}$,

$$\mu^+(K \cap \xi^\perp) = \int_{K\cap\xi^\perp} f = \int_{S^{n-1}\cap\xi^\perp} \left(\int_0^{\rho_K(\theta)} r^{n-2} f(r\theta)\ dr \right) d\theta = \int_{S^{n-1}\cap\xi^\perp} F(\theta) d\theta,$$

where

$$F(\theta) = \int_0^{\rho_K(\theta)} r^{n-2} f(r\theta)\ dr \qquad\qquad (\theta \in S^{n-1}).$$

Therefore, inequality (6) can be written in terms of the spherical Radon transform

$$RF(\xi) \le \max_{\theta\in S^{n-1}} \mu^+(K \cap \theta^\perp) \tag{7}$$

for all $\xi \in S^{n-1}$. Note that the right-hand side of (7) does not depend on ξ.

Let D be an intersection body such that the distance $d_{\mathrm{ovr}}(K, \mathcal{I}_n)$ is almost realized, i.e. $K \subset D$ and for some small $\delta > 0$,

$$|D|^{1/n} \leq (1+\delta) d_{\mathrm{ovr}}(K, \mathcal{I}_n) |K|^{1/n}. \tag{8}$$

Integrating both sides of (7) by ξ over the sphere with respect to the measure ν_D corresponding to D by definition (5), we get

$$\int_{S^{n-1}} \rho_D(\theta) \left(\int_0^{\rho_K(\theta)} r^{n-2} f(r\theta) dr \right) d\theta \leq \nu_D(S^{n-1}) \max_{\theta \in S^{n-1}} \mu^+(K \cap \theta^\perp). \tag{9}$$

The left-hand side of (9) is equal to

$$\int_{S^{n-1}} \left(\int_0^{\rho_K(\theta)} (\rho_D(\theta) - r) r^{n-2} f(r\theta) dr \right) d\theta \ + \ \int_{S^{n-1}} \left(\int_0^{\rho_K(\theta)} r^{n-1} f(r\theta) dr \right) d\theta$$

$$\geq \int_{S^{n-1}} \left(\int_0^{\rho_K(\theta)} r^{n-1} f(r\theta) dr \right) d\theta = \int_K f = \mu(K), \tag{10}$$

because $K \subset D$ implies $\rho_D(\theta) \geq \rho_K(\theta)$ for every θ.

Now we estimate the left-hand side of (9) from above. We use $R1(\xi) = |S^{n-2}|$ for every $\xi \in S^{n-1}$, definition (5), Hölder's inequality and a standard formula for volume:

$$\nu_D(S^{n-1}) = \frac{1}{|S^{n-2}|} \int_{S^{n-1}} R1(\xi) d\nu_D(\xi) = \frac{1}{|S^{n-2}|} \int_{S^{n-1}} \rho_D(\xi) d\xi$$

$$\leq \frac{|S^{n-1}|^{\frac{n-1}{n}}}{|S^{n-2}|} \left(\int_{S^{n-1}} \rho_D^n(\xi) d\xi \right)^{\frac{1}{n}} \leq 2|D|^{\frac{1}{n}}.$$

By using (8), sending $\delta \to 0$, and combining the estimates above, we obtain the conclusion of the theorem. Note that the uniform measure on the sphere was not normalized in the calculations. □

3 A counterexample

We move on to the proof of Theorem 1.3. We may clearly assume that the dimension n exceeds a given universal constant C, as otherwise the conclusion of the theorem is trivial. We shall need the following well-known Bernstein-type inequality. A proof is provided for completeness:

Lemma 3.1. *Let $Y_1, \ldots, Y_N$ be independent, identically-distributed random variables attaining values in the interval $[0,1]$. Let $p \in [0,1]$ satisfy $p \geq \mathbb{E}Y_1$. Then,*

$$\mathbb{P}\left(\frac{1}{N} \sum_{i=1}^N Y_i \geq 3p \right) \leq e^{-pN}.$$

Proof. Since $Y_1 \in [0,1]$ with $\mathbb{E}Y_1 \le p$,

$$\mathbb{E}e^{Y_1} = 1 + \sum_{q=1}^{\infty} \frac{\mathbb{E}Y_1^{q-1}Y_1}{q!} \le 1 + \sum_{q=1}^{\infty} \frac{\mathbb{E}Y_1}{q!} \le 1 + p(e-1).$$

Therefore,

$$\mathbb{E}e^{\sum_{i=1}^N Y_i} = \left(\mathbb{E}e^{Y_1}\right)^N \le (1+p(e-1))^N \le e^{Np(e-1)}.$$

By the Markov-Chebyshev inequality,

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^N Y_i \ge 3p\right) \le e^{-3Np}\mathbb{E}e^{\sum_{i=1}^N Y_i} \le e^{Np(e-4)} < e^{-pN}. \qquad \square$$

For $t \in \mathbb{R}$ we set $\varphi(t) = e^{-t^2/2}$. Later on, we will apply Lemma 3.1 for $Y_i = \varphi(t + R\Theta_i \cdot \xi)$, where Θ_i is a random point in the sphere S^{n-1} and ξ is a fixed unit vector.

Lemma 3.2. *Let $n \ge 4$ and let $\Theta \in S^{n-1}$ be a random point, distributed uniformly over S^{n-1}. Let $\xi \in S^{n-1}$ be a fixed unit vector. Then for any $t \in \mathbb{R}$ and $R > \sqrt{n}$,*

$$\mathbb{E}\varphi\left(t + R\Theta \cdot \xi\right) \le C\frac{\sqrt{n}}{R} \cdot \varphi\left(\frac{c\sqrt{n}}{R}t\right),$$

where $C > 0$ and $0 < c < 1$ are universal constants.

Proof. Denote $Z = \Theta \cdot \xi$. Then Z is a random variable supported in the interval $[-1,1]$ whose density in this interval is proportional to the function $s \mapsto (1-s^2)^{(n-3)/2}$. Setting $k = n-3$ we see that we need to prove that

$$\alpha_k \int_{-1}^{1} \varphi(Rs+t) \cdot \left(1-s^2\right)^{k/2} ds \le C\frac{\sqrt{k}}{R} \cdot e^{-\frac{kt^2}{2R^2}}, \tag{11}$$

where

$$\alpha_k^{-1} = \int_{-1}^{1} \left(1-s^2\right)^{k/2} ds \ge \int_{-1/\sqrt{k}}^{1/\sqrt{k}} \left(1-s^2\right)^{k/2} ds \ge \frac{c}{\sqrt{k}}. \tag{12}$$

In order to prove (11), we note that

$$\alpha_k \int_{-1}^{1} \varphi(Rs+t) \cdot \left(1-s^2\right)^{k/2} ds \le C\sqrt{k}\int_{-\infty}^{\infty} e^{-(Rs+t)^2/2 - ks^2/2} ds = C\sqrt{\frac{2\pi k}{R^2+k}} \cdot e^{-\frac{kt^2}{2(R^2+k)}},$$

where we used (12) and the inequality $(1-\alpha)^m \le \exp(-\alpha m)$, valid for all $0 < \alpha < 1$ and $m > 0$. Thus (11) is proven. $\square$

By combining Lemma 3.1 and Lemma 3.2 we obtain the following:

Corollary 3.3. *Let $N \geq n \geq 4$ and let $\Theta_1, \ldots, \Theta_N \in S^{n-1}$ be independent, identically-distributed random vectors, distributed uniformly over the sphere S^{n-1}. Fix $\xi \in S^{n-1}, t \in \mathbb{R}, R > \sqrt{n}$ and $\alpha > 0$. Then,*

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N}\varphi\left(t + R\xi \cdot \Theta_i\right) \geq C \max\left\{\alpha, \frac{\sqrt{n}}{R} \cdot \varphi\left(\frac{c\sqrt{n}}{R}t\right)\right\}\right) \leq e^{-N\alpha},$$

where $C > 0$ and $0 < c < 1$ are universal constants.

Proof. Set $Y_i = \varphi(t + R\xi \cdot \Theta_i)$. Then $Y_1, \ldots, Y_N$ are independent, identically-distributed random variables attaining values in the interval $[0,1]$. Set

$$p = \max\left\{\alpha, C\frac{\sqrt{n}}{R} \cdot \varphi\left(\frac{c\sqrt{n}}{R}t\right)\right\},$$

where $c, C > 0$ are the constants from Lemma 3.2. Then $\mathbb{E}Y_1 \leq p$, according to Lemma 3.2. By Lemma 3.1,

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i \geq 3p\right) \leq \exp(-Np) \leq \exp(-N\alpha),$$

where we used that $p \geq \alpha$ in the last passage. □

The function $\varphi(s) = e^{-s^2/2}$ has a bounded derivative $\varphi'(s) = -se^{-s^2/2}$. Therefore φ is a 1-Lipschitz function on the entire real line. This Lipschitz property enables us to make the estimate of Corollary 3.3 uniform in $\xi \in S^{n-1}$, as explained in the following:

Proposition 3.4. *Assume that $n \geq 5$, that $N \geq 10n \log n$ and that $\sqrt{n} \leq R \leq n$. Let $\Theta_1, \ldots, \Theta_N$ be independent, identically-distributed random vectors, distributed uniformly in S^{n-1}. Then with probability of at least $1 - n^{-n}$ the following holds: For all $\xi \in S^{n-1}$ and $t \in \mathbb{R}$,*

$$\frac{1}{N}\sum_{i=1}^{N}\varphi\left(t + R\xi \cdot \Theta_i\right) \leq \frac{Cn\log n}{\min\{N, n^3\}} + C\frac{\sqrt{n}}{R} \cdot \varphi\left(\frac{c\sqrt{n}}{R}t\right), \tag{13}$$

where $c, C > 0$ are universal constants.

Proof. For all possible choices of $\theta_1, \ldots, \theta_N \in S^{n-1}$, the function

$$G_{\theta_1,\ldots,\theta_N}(t, \xi) = \frac{1}{N}\sum_{i=1}^{N}\varphi(t + R\xi \cdot \theta_i) \qquad (t \in \mathbb{R}, \xi \in S^{n-1})$$

is a Lipschitz function on $\mathbb{R} \times S^{n-1}$ whose Lipschitz constant is at most $R + 1 \leq n + 1$. Set $\delta = n^{-3}$, and let $\mathcal{F} \subseteq S^{n-1}$ be a δ-net, i.e., for any $x \in S^{n-1}$ there exists $y \in \mathcal{F}$

with $|x-y| \leq \delta$. By a standard volumetric argument (see, e.g., [22]), there exists a δ-net $\mathcal{F} \subseteq S^{n-1}$ with cardinality

$$\#(\mathcal{F}) \leq \left(\frac{5}{\delta}\right)^n \leq e^{6n\log n}. \tag{14}$$

Let $I \subseteq \mathbb{R}$ be the set of all integer multiples of n^{-3} that lie in the interval $[-n^3, n^3]$. Then for any $\xi \in S^{n-1}$ and $t \in [-n^3, n^3]$ there exists $\tilde{\xi} \in \mathcal{F}$ and $\tilde{t} \in I$ with

$$G_{\theta_1,\ldots,\theta_N}(t,\xi) \leq (n+1)\cdot\frac{2}{n^3} + G_{\theta_1,\ldots,\theta_N}(\tilde{t},\tilde{\xi}), \tag{15}$$

by the aforementioned Lipschitz property of $G_{\theta_1,\ldots,\theta_N}$. Let us now apply Corollary 3.3 with $\alpha = 10(n\log n)/N$, to obtain

$$\begin{aligned}\mathbb{P}\left(\forall\xi \in \mathcal{F}, t \in I, \quad G_{\Theta_1,\ldots,\Theta_N}(t,\xi) \leq C\max\left\{\alpha, \frac{\sqrt{n}}{R}\cdot\varphi\left(\frac{c\sqrt{n}}{R}t\right)\right\}\right) \\ \geq 1 - \#(\mathcal{F})\cdot\#(I)\cdot e^{-N\alpha} \geq 1 - n^7\cdot e^{6n\log n}\cdot e^{-10n\log n} \geq 1 - n^{-n}.\end{aligned} \tag{16}$$

The constant c in (16) is at most one. Hence for any $t, \tilde{t} \in [-n^3, n^3]$ with $|t-\tilde{t}| \leq n^{-3}$ we have that $\varphi(c\sqrt{n}\cdot\tilde{t}/R) \leq 5\varphi(c\sqrt{n}\cdot t/R)$. We therefore conclude from (15) and (16) that

$$\mathbb{P}\left(\forall\xi \in S^{n-1}, |t| \leq n^3, \quad G_{\Theta_1,\ldots,\Theta_N}(t,\xi) \leq \frac{Cn\log n}{N} + \frac{\tilde{C}\sqrt{n}}{R}\cdot\varphi\left(\frac{c\sqrt{n}}{R}t\right)\right) \geq 1 - n^{-n}.$$

We have thus proven that with probability of at least $1-n^{-n}$, inequality (13) holds true for all $\xi \in S^{n-1}$ and $|t| \leq n^3$. The validity of (13) when $|t| > n^3$ is much easier, as in this case

$$\frac{1}{N}\sum_{i=1}^N \varphi\left(t + R\xi\cdot\Theta_i\right) \leq \frac{1}{N}\sum_{i=1}^N \varphi\left(|t| - n\right) \leq \varphi(n^3-n) \leq e^{-10n} \leq \frac{n\log n}{\min\{N, n^3\}},$$

with probability one. This completes the proof. □

Remark 3.5. The use of the δ-net in the proof of Proposition 3.4 is probably not optimal, and it is the reason for the appearance of the log log-factor in our result. We suspect that this factor may be improved or eliminated by using more sophisticated tools from the theory of Gaussian processes, such as Slepian's lemma, Talgrand's majorizing measure or related results. In fact, perhaps Gordon's minmax theorem may be used in order to improve the double logarithm to a triple logarithm in Theorem 1.1. These considerations will be expanded upon elsewhere.

By iterating Proposition 3.4 twice we obtain the following:

Lemma 3.6. *Assume that* $n \geq C_1$ *and let us set*

$$N_1 = n^3, \quad N_2 = \lceil n \log^3 n \rceil, \quad R_1 = n/\sqrt{\log n} \quad \text{and} \quad R_2 = n/\sqrt{\log\log n}. \qquad (17)$$

Then there exist unit vectors $\theta_1, \ldots, \theta_{N_1}, \eta_1, \ldots, \eta_{N_2} \in S^{n-1}$ *such that for all* $\xi \in S^{n-1}$ *and* $t \in \mathbb{R}$,

$$\frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \varphi\left(t + R_1 \xi \cdot \theta_i + R_2 \xi \cdot \eta_j\right) \leq \frac{C}{\sqrt{n} \log n} + C \frac{\sqrt{n}}{R_2} \cdot \varphi\left(c \frac{\sqrt{n}}{R_2} t\right),$$

where $c, C, C_1 > 0$ *are universal constants.*

Proof. We may assume that $n \geq 5$ is sufficiently large so that $cR_2/R_1 > 1$, where throughout this proof $c > 0$ is the constant from Proposition 3.4. By the conclusion of Proposition 3.4, we may fix unit vectors $\theta_1, \ldots, \theta_{N_1} \in S^{n-1}$ such that for all $\xi \in S^{n-1}$ and $s \in \mathbb{R}$,

$$\frac{1}{N_1} \sum_{i=1}^{N_1} \varphi\left(s + R_1 \xi \cdot \theta_i\right) \leq \frac{Cn \log n}{N_1} + C \frac{\sqrt{n}}{R_1} \cdot \varphi\left(\frac{c\sqrt{n}}{R_1} s\right). \qquad (18)$$

Note that $Cn(\log n)/N_1 \leq C/n$. In particular, for any choice of $\eta_1, \ldots, \eta_{N_2} \in S^{n-1}$, and for any $\xi \in S^{n-1}$ and $t \in \mathbb{R}$,

$$\frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \varphi\left(t + R_1 \xi \cdot \theta_i + R_2 \xi \cdot \eta_j\right) \leq \frac{C}{n} + C \frac{\sqrt{n}}{R_1} \cdot \frac{1}{N_2} \sum_{j=1}^{N_2} \varphi\left(\frac{c\sqrt{n}}{R_1} t + \frac{cR_2\sqrt{n}}{R_1} \xi \cdot \eta_j\right),$$

where we used (18) with $s = t + R_2 \xi \cdot \eta_j$. Let us now apply Proposition 3.4 with

$$R = cR_2\sqrt{n}/R_1 > \sqrt{n} \qquad \text{and} \qquad N = N_2 \geq n \log^3 n \geq 10 n \log n.$$

From the conclusion of this proposition we obtain that there exist unit vectors $\eta_1, \ldots, \eta_{N_2} \in S^{n-1}$ such that for all $\xi \in S^{n-1}$ and $t \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{N_2} \sum_{j=1}^{N_2} \varphi\left(\frac{c\sqrt{n}}{R_1} t + \frac{cR_2\sqrt{n}}{R_1} \xi \cdot \eta_j\right) &\leq \frac{Cn \log n}{N_2} + C \frac{\sqrt{n}}{R} \cdot \varphi\left(\frac{c\sqrt{n}}{R} \cdot \frac{c\sqrt{n}}{R_1} t\right) \\ &\leq \frac{\tilde{C}}{\log^2 n} + C \frac{\sqrt{n}}{R} \cdot \varphi\left(\frac{c\sqrt{n}}{R_2} t\right). \end{aligned}$$

We may combine the two inequalities above to obtain that for all $\xi \in S^{n-1}$ and $t \in \mathbb{R}$,

$$\frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \varphi\left(t + R_1 \xi \cdot \theta_i + R_2 \xi \cdot \eta_j\right) \leq \frac{\tilde{C}}{\sqrt{n} \log^{3/2} n} + C \frac{\sqrt{n}}{R_2} \cdot \varphi\left(\frac{c\sqrt{n}}{R_2} t\right). \qquad \square$$

The reason we iterated Proposition 3.4 only twice and not thrice or more in the proof of Lemma 3.6 is basically the non-optimal use of the δ-net alluded to in Remark 3.5.

Write $e_1, \ldots, e_n$ for the standard unit vectors in $\mathbb{R}^n$. Let $\theta_1, \ldots, \theta_{N_1}$ and $\eta_1, \ldots, \eta_{N_2}$ be the unit vectors whose existence is proven in Lemma 3.6. We now define the centrally-symmetric convex body $K \subseteq \mathbb{R}^n$ to be the convex hull of the $2(N_1 + N_2 + n)$ vectors

$$\pm R_1\theta_1, \ldots, \pm R_1\theta_{N_1}, \quad \pm R_2\eta_1, \ldots, \pm R_2\eta_{N_2}, \quad \pm ne_1, \ldots, \pm ne_n.$$

We write γ_n for the standard Gaussian measure in $\mathbb{R}^n$, whose density is

$$x \mapsto (2\pi)^{-n/2} \exp(-|x|^2/2).$$

We will use below the standard bound $(2\pi)^{-1/2} \int_t^\infty \varphi(s)ds \leq \varphi(t)/2$ for all $t \geq 0$.

Lemma 3.7. *We have that $|K| \leq C^n$, where $C > 0$ is a universal constant.*

Proof. We will mimick an argument by Gluskin [7]. For a centrally-symmetric convex body $K \subseteq \mathbb{R}^n$ we denote its polar body by

$$K^\circ = \{x \in \mathbb{R}^n \, ; \, \forall y \in K, \, x \cdot y \leq 1\} .$$

Let Z be a standard Gaussian random vector in $\mathbb{R}^n$. According to the Khatri-Sidak lemma ([10], [23], see also [7] for a simple proof),

$$\begin{aligned}\mathbb{P}(Z \in 5nK^\circ) &= \mathbb{P}\left(\forall i,j,k, \quad |R_1 Z \cdot \theta_i| \leq 5n, \ |R_2 Z \cdot \eta_j| \leq 5n \text{ and } |nZ \cdot e_k| \leq 5\right) \\ &\geq \prod_{i=1}^{N_1} \mathbb{P}\left(|R_1 Z \cdot \theta_i| \leq 5n\right) \cdot \prod_{j=1}^{N_2} \mathbb{P}\left(|R_2 Z \cdot \eta_j| \leq 5n\right) \cdot \prod_{k=1}^{n} \mathbb{P}\left(|nZ \cdot e_k| \leq 5n\right).\end{aligned}$$

Since $Z \cdot \theta_i$ is a standard Gaussian random variable, we know that

$$\mathbb{P}\left(|R_1 Z \cdot \theta_i| \leq 5n\right) = 1 - \frac{2}{\sqrt{2\pi}} \int_{5n/R_1}^\infty \varphi(s)ds \geq 1 - \varphi(5n/R_1).$$

Consequently,

$$\gamma_n(5nK^\circ) = \mathbb{P}(Z \in 5nK^\circ) \geq \left(1 - \varphi(5n/R_1)\right)^{N_1} \cdot \left(1 - \varphi(5n/R_2)\right)^{N_2} \cdot \left(1 - \varphi(5)\right)^n.$$

Recalling from (17) the values of our parameters, we obtain

$$\gamma_n(5nK^\circ) \geq \left(1 - \frac{1}{n^{10}}\right)^{n^3} \cdot \left(1 - \frac{1}{(\log n)^{10}}\right)^{1 + n\log^3 n} \cdot c^n \geq e^{-\tilde{c}n}.$$

Since the density of the measure γ_n does not exceed $(2\pi)^{-n/2}$, we conclude that

$$|5nK^\circ| > c^n.$$

The conclusion of the lemma now follows from the Santalo inequality (see e.g. [2, Theorem 1.3.4]). $\square$

Since $e_1, \ldots, e_n \in K/n$ while $K \subseteq \mathbb{R}^n$ is convex and centrally-symmetric, we know that

$$K \supseteq \sqrt{n} B^n. \tag{19}$$

Recall that $\int_{\mathbb{R}^n} |x|^2 d\gamma_n(x) = n$. By the Markov-Chebyshev inequality,

$$\gamma_n \left(2\sqrt{n} B^n \right) \geq 1/2. \tag{20}$$

Let us now define the probability measure μ to be the convolution

$$\mu = \gamma_n * \left(\frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{\delta_{R_1\theta_i + R_2\eta_j} + \delta_{-R_1\theta_i - R_2\eta_j}}{2} \right)$$

where δ_y is the delta measure at the point $y \in \mathbb{R}^n$. Then μ is a probability measure in $\mathbb{R}^n$.

Lemma 3.8. *$\mu(4K) \geq 1/2$.*

Proof. For any i and j, the convex body $4K = 2K + 2K$ contains the set $\pm R_1\theta_i \pm R_2\eta_j + 2\sqrt{n}B^n$, according to (19). The measure μ is an average of translates of γ_n, each centered at a point of the form $\pm R_1\theta_i \pm R_2\eta_j$. Consequently,

$$\mu(4K) \geq \gamma_n(2\sqrt{n}B^n) \geq \frac{1}{2}. \qquad \square$$

Write g for the continuous density of the measure μ. Setting $\varphi_n(x) = \exp(-|x|^2/2)$ for $x \in \mathbb{R}^n$, we have

$$g(x) = \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{\varphi_n(x + R_1\theta_i + R_2\eta_j) + \varphi_n(x - R_1\theta_i - R_2\eta_j)}{2 \cdot (2\pi)^{n/2}}.$$

Note that for any $\xi \in S^{n-1}, z \in \mathbb{R}^n$ and $t \geq 0$,

$$\int_{t\xi + \xi^\perp} \frac{\varphi_n(x+z)}{(2\pi)^{n/2}} dx = \frac{\varphi(t + z \cdot \xi)}{\sqrt{2\pi}}. \tag{21}$$

Lemma 3.9. *For any $\xi \in S^{n-1}$ and $t \geq 0$,*

$$\int_{t\xi + \xi^\perp} g \leq C \frac{\sqrt{\log \log n}}{\sqrt{n}}.$$

Proof. By (21) we have that

$$\int_{t\xi + \xi^\perp} g = \frac{1}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=2}^{N_2} \frac{\varphi\left(t + R_1\xi \cdot \theta_i + R_2\xi \cdot \eta_j\right) + \varphi\left(-t + R_1\xi \cdot \theta_i + R_2\xi \cdot \eta_j\right)}{2 \cdot \sqrt{2\pi}}.$$

Therefore, according to Lemma 3.6 and as $\varphi \leq 1$,

$$\int_{t\xi + \xi^\perp} g \leq \frac{C}{\sqrt{n}\log n} + C\frac{\sqrt{n}}{R_2} \leq \tilde{C} \frac{\sqrt{\log \log n}}{\sqrt{n}},$$

completing the proof. $\square$

Proof of Theorem 1.3. We set

$$f(x) = g(x) \cdot 1_{4K}(x)/\mu(4K)$$

where 1_{4K} is the function that equals one on $4K$ and vanishes otherwise. Then f is an even, continuous probability density supported on $T = 4K$. According to Lemma 3.8,

$$f(x) \leq 2g(x) \qquad \text{for } x \in \mathbb{R}^n.$$

Therefore, from Lemma 3.9, for any $\xi \in S^{n-1}$ and $t \geq 0$,

$$\int_{T\cap(\xi^\perp+t\xi)} f(y)dy \leq 2\int_{\xi^\perp+t\xi} g(y)dy \leq \frac{C\sqrt{\log\log n}}{\sqrt{n}}. \tag{22}$$

We also know that $|T|^{1/n} \leq C$, according to Lemma 3.7. Therefore the desired estimate (2) follows from (22). □

The left-hand side inequality in Theorem 1.1 clearly follows from Theorem 1.3. Note also that Theorem 1.3 entails the optimality, up to a factor of $\log\log n$, of Corollary 1 from [19].

4 Measures admitting tail bounds

This section is devoted to the proof of Theorem 1.5, which is a modification of the proof of Theorem 1.3. We are given a dimension n and $\alpha \in (0, 2]$. We may assume that $n \geq 10$ as otherwise the conclusion of the theorem is trivial.

Lemma 4.1. *Let $\Theta \in S^{n-1}$ be a random vector, distributed uniformly over S^{n-1}. Let $\xi \in S^{n-1}$ be a fixed unit vector. Then,*

$$\mathbb{P}\left(|\langle\Theta,\xi\rangle| \geq \frac{1}{\sqrt{n}}\right) \geq c \qquad \textit{and} \qquad \mathbb{E}e^{\left(\frac{\sqrt{n}|\langle\Theta,\xi\rangle|+1}{2}\right)^\alpha} \leq C_1,$$

where $c, C_1 > 0$ are universal constants.

Proof. Denote $Z = \langle\Theta,\xi\rangle$. As in the proof of Lemma 3.2, the density of Z in the interval $[-1, 1]$ is proportional to $\beta_n(1-t^2)^{(n-3)/2}$, where β_n satisfies $c\sqrt{n} \leq \beta_n \leq C\sqrt{n}$. We need to prove that

$$\sqrt{n}\int_{-1}^{1} 1_{\{|s|\geq 1/\sqrt{n}\}}\left(1-s^2\right)^{\frac{n-3}{2}} ds \geq c \quad \text{and} \quad \sqrt{n}\int_{-1}^{1} e^{\left(\frac{\sqrt{n}|s|+1}{2}\right)^\alpha}\left(1-s^2\right)^{\frac{n-3}{2}} ds \leq C_1 \tag{23}$$

The left-hand side inequality in (23) follows from

$$\int_{-1}^{1} 1_{\{|s|\geq 1/\sqrt{n}\}}\left(1-s^2\right)^{\frac{n-3}{2}} ds \geq 2\int_{1/\sqrt{n}}^{2/\sqrt{n}}\left(1-s^2\right)^{n/2} ds \geq \frac{2}{\sqrt{n}} \cdot (1-4/n)^{n/2} \geq \frac{c}{\sqrt{n}}.$$

As for the right-hand side inequality in (23), we argue as follows:

$$\sqrt{n}\int_{-1}^{1} e^{\left(\frac{\sqrt{n}|s|+1}{2}\right)^{\alpha}} \left(1-s^2\right)^{\frac{n-3}{2}} ds \leq \sqrt{n}\int_{-\infty}^{\infty} e^{\left(\frac{\sqrt{n}|s|+1}{2}\right)^{\alpha}} e^{-s^2 n/3} ds = \int_{-\infty}^{\infty} e^{\left(\frac{|t|+1}{2}\right)^{\alpha}} e^{-t^2/3} dt$$

$$\leq 14 e^{4^{\alpha}} + 2\int_{7}^{\infty} e^{\left(\frac{t+1}{2}\right)^{\alpha}} e^{-t^2/3} dt \leq 14 e^{16} + 2\int_{0}^{\infty} e^{\left(\frac{4t}{7}\right)^{2}} e^{-t^2/3} dt = 14\cdot e^{16} + \sqrt{147\pi}. \qquad \square$$

Let us now introduce the parameter

$$N = \lceil n^8 \cdot \exp(8 n^{\alpha/2}) \rceil.$$

Let $\Theta_1, \ldots, \Theta_N$ be independent random vectors, distributed uniformly in S^{n-1}.

Lemma 4.2. *Assume that $n \geq \bar{C}$. Then with probability of at least $1 - e^{-n^2}$ of selecting $\Theta_1, \ldots, \Theta_N$ the following holds: For all $\xi \in S^{n-1}$,*

$$\frac{1}{N}\sum_{i=1}^{N} |\langle \Theta_i, \xi \rangle| \geq \frac{c}{\sqrt{n}} \qquad \text{and} \qquad \frac{1}{N}\sum_{i=1}^{N} e^{\left(\sqrt{n}|\langle \Theta_i, \xi\rangle|/2\right)^{\alpha}} \leq C.$$

Here, $c, C, \bar{C} > 0$ are universal constants.

Proof. The constant $\bar{C} > 10$ will be a sufficiently large universal constant whose value will be determined later on. Fix $\xi \in S^{n-1}$ and set

$$Y_i = 1_{\{|\langle \Theta_i, \xi \rangle| \geq 1/\sqrt{n}\}} \qquad (i = 1, \ldots, N).$$

Then $Y_1, \ldots, Y_N$ are independent, identically-distributed random variables attaining values in $\{0, 1\}$, with $\mathbb{P}(Y_i = 1) \geq c$ according to Lemma 4.1. By a standard estimate for the binomial distribution (see, e.g., [1, Chapter 2]),

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i \geq c/2\right) \geq 1 - C_2 e^{-c_1 N}.$$

Consequently, for any fixed $\xi \in S^{n-1}$,

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} |\langle \Theta_i, \xi\rangle| \geq \frac{c_2}{\sqrt{n}}\right) \geq \mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i \geq c/2\right) \geq 1 - C_2 e^{-c_1 N}. \tag{24}$$

Next, set

$$Z_i = \exp\left(\left(\frac{\sqrt{n}|\langle \Theta_i, \xi\rangle| + 1}{2}\right)^{\alpha}\right) \qquad (i = 1, \ldots, N).$$

Then $1 \leq Z_i \leq \exp(n^{\alpha/2}) \leq \sqrt{N}$ while $\mathbb{E} Z_i \leq C_3$ according to Lemma 4.1. We may thus use Lemma 3.1 and conclude that for any fixed $\xi \in S^{n-1}$,

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} Z_i \leq 3C_3\right) = \mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} \frac{Z_i}{\sqrt{N}} \leq \frac{3C_3}{\sqrt{N}}\right) \geq 1 - e^{-NC_3/\sqrt{N}} = 1 - e^{-C_3\sqrt{N}}. \tag{25}$$

We now select the universal constant $\bar{C} > 10$ large enough so that the assumption $n \geq \bar{C}$ implies

$$C_2 e^{-c_1 n^8} + e^{-C_3 n^4} \leq e^{-50n^2} \qquad \text{and} \qquad \frac{\sqrt{n}}{c_2} \leq \frac{n^2}{2} \tag{26}$$

where c_1, c_2, C_2 are the constants from (24) while C_3 is the constant from (25). Consider the functions

$$f(t) = t \qquad \text{and} \qquad g(t) = \exp\left\{ \left(\frac{\sqrt{n}t + 1}{2} \right)^\alpha \right\} \qquad \text{for } 0 \leq t \leq 1.$$

Clearly f is a 1-Lipschitz function, while for any $0 \leq t \leq 1$,

$$|g'(t)| = g(t) \cdot \frac{\alpha\sqrt{n}}{2} \cdot \left(\frac{\sqrt{n}t + 1}{2} \right)^{\alpha - 1} \leq 2\sqrt{n} \cdot \exp\left\{ 2\left(\frac{\sqrt{n}t + 1}{2} \right)^\alpha \right\} \leq n e^{2n^{\alpha/2}} \leq N^{1/4},$$

where we used the elementary inequality $x^{\alpha-1}e^{x^\alpha} \leq 2e^{2x^\alpha}$ with $x = (\sqrt{n}t + 1)/2 \geq 1/2$. Hence g is an $N^{1/4}$-Lipschitz function on S^{n-1}. Set $\delta = N^{-1/3}$ and let $\mathcal{F} \subseteq S^{n-1}$ be a δ-net of cardinality

$$\#(\mathcal{F}) \leq \left(\frac{5}{\delta} \right)^n \leq e^{6n \log(N^{1/3})} = N^{2n} \leq (ne^n + 1)^{16n} \leq e^{40n^2}.$$

From (24), (25) and (26), with probability of at least $1 - e^{-10n^2}$ of selecting $\Theta_1, \ldots, \Theta_N$, for all $\xi \in \mathcal{F}$,

$$\frac{1}{N}\sum_{i=1}^N |\langle \Theta_i, \xi \rangle| \geq \frac{c_2}{\sqrt{n}} \qquad \text{and} \qquad \frac{1}{N}\sum_{i=1}^N \exp\left\{ \left(\frac{\sqrt{n}|\langle \Theta_i, \xi \rangle| + 1}{2} \right)^\alpha \right\} \leq 3C_3. \tag{27}$$

Since $\mathcal{F}$ is a δ-net, from (27) and the Lipschitz properties of f and g we obtain that with probability of at least $1 - e^{-10n^2}$, for all $\xi \in S^{n-1}$,

$$\frac{1}{N}\sum_{i=1}^N |\langle \Theta_i, \xi \rangle| \geq \frac{c_2}{\sqrt{n}} - \delta \qquad \text{and} \qquad \frac{1}{N}\sum_{i=1}^N \exp\left\{ \left(\frac{\sqrt{n}|\langle \Theta_i, \xi \rangle| + 1}{2} \right)^\alpha \right\} \leq 3C_3 + N^{1/4}\delta. \tag{28}$$

However, $\delta = N^{-1/3} \leq \frac{1}{n^2} \leq \frac{c_2}{2\sqrt{n}}$ according to (26), while $N^{1/4}\delta \leq 1$. Hence the conclusion of the lemma follows from (28). □

We define the centrally-symmetric convex body $K \subseteq \mathbb{R}^n$ to be the convex hull of the $2N + 2n$ points

$$\pm R\Theta_1, \ldots, \pm R\Theta_N, \pm ne_1, \ldots, \pm ne_n,$$

where

$$R = n^{1-\alpha/4}.$$

From now on in this paper, we write $c, C, C_1, \tilde{C}$ etc. for various positive constants that depend solely on $\alpha \in (0, 2]$.

Lemma 4.3. *Assume that $n \geq \bar{C}$. Then with probability one, $|K| \leq C^n$, where $C, \bar{C} > 0$ depend solely on α.*

Proof. As in the proof of Corollary 4.3, it suffices to show that $\gamma_n(10nK^\circ) \geq c^n$. Let Z be a standard Gaussian random vector in $\mathbb{R}^n$, independent of the Θ_i's. By the Khatri-Sidak lemma,

$$\gamma_n(10nK^\circ) = \mathbb{P}\left(\forall i, j \quad |\langle Z, \Theta_i \rangle| \leq 10n/R \ \text{ and } \ |\langle Z, e_j \rangle| \leq 10\right)$$
$$\geq (1 - \varphi(10n/R))^N \cdot (1 - \varphi(10))^n \geq \left(1 - e^{-50n^{\alpha/2}}\right)^{n^8 \exp(8n^{\alpha/2}) + 1} \cdot (1 - \exp(-c))^n.$$

However, $n^8 \exp(8n^{\alpha/2}) + 1 \leq C \cdot e^{10n^{\alpha/2}}$ where $C > 0$ depends solely on α. Moreover, $e^{10n^{\alpha/2}} \geq 2$ assuming that $n \geq \bar{C}$ for some $\bar{C} > 0$ depending on α. Hence,

$$\gamma_n(10nK^\circ) \geq \left(1 - e^{-50n^{\alpha/2}}\right)^{C \cdot e^{10n^{\alpha/2}}} \cdot c^n \geq \tilde{c}^n,$$

where $\tilde{c} > 0$ depends only on α. □

We define the probability measure μ to be the convolution $\mu = \gamma_n * \nu$ where

$$\nu = \frac{1}{N} \sum_{i=1}^{N} \frac{\delta_{R\Theta_i} + \delta_{-R\Theta_i}}{2}.$$

Lemma 4.4. *Assume that $n \geq \bar{C}$. Then with probability one, $\mu(3K) \geq 1 - C\exp(-cn) \geq 1/2$.*

Proof. Note that $K \supseteq \sqrt{n} B^n$ as $\pm n e_1, \ldots, \pm n e_n \in K$. Consequently, the convex body $3K$ contains the set $\pm R\Theta_i + 2\sqrt{n} B^n$ for any $i = 1, \ldots, N$. As in the proof of Lemma 3.8, the measure μ is a mixture of translates of γ_n, each centered at a point of the form $\pm R\Theta_i$. Therefore $\mu(3K)$ is at least $\gamma_n(2\sqrt{n} B^n)$ which in turn is at least $1 - C\exp(-cn) \geq 1/2$ by a standard estimate. □

Lemma 4.5. *Assume that $n \geq \bar{C}$. Then with probability of at least $1 - e^{-n^2}$, the measure μ admits ψ_α-tails with parameters (C, c). Here, $c, C, \bar{C} > 0$ depend only on α.*

Proof. We will assume that the event described in Lemma 4.2 holds true, which happens with probability of at least $1 - e^{-n^2}$. We need to show that for any $\xi \in S^{n-1}$, setting $E_\xi = \int_{\mathbb{R}^n} |\langle x, \xi \rangle| d\mu(x)$, we have

$$\mu\left(\{x \in \mathbb{R}^n \, ; \, |\langle \xi, x \rangle| \geq tE_\xi\}\right) \leq C\exp(-ct^\alpha). \qquad \text{for } t > 0. \tag{29}$$

Since the event described in Lemma 4.2 holds true, we know that for any $\xi \in S^{n-1}$,

$$E_\xi = \int_{\mathbb{R}^n} |\langle x, \xi \rangle| d\mu(x) = \frac{1}{N} \sum_{i=1}^N \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{|t - \langle \xi, R\Theta_i \rangle| + |t + \langle \xi, R\Theta_i \rangle|}{2} e^{-t^2/2} dt$$
$$\geq \frac{1}{N} \sum_{i=1}^N |\langle \xi, R\Theta_i \rangle| \geq c \frac{R}{\sqrt{n}}.$$

Therefore,

$$\int_{\mathbb{R}^n} \exp\left\{ \left(c_1 \frac{|\langle \xi, x \rangle|}{E_\xi} \right)^\alpha \right\} d\nu(x) = \frac{1}{N} \sum_{i=1}^N \exp\left\{ \left(c_1 \frac{|\langle \xi, R\Theta_i \rangle|}{E_\xi} \right)^\alpha \right\}$$
$$\leq \frac{1}{N} \sum_{i=1}^N \exp\left\{ \left(\frac{\sqrt{n} |\langle \xi, \Theta_i \rangle|}{2} \right)^\alpha \right\} \leq C. \tag{30}$$

A standard application of the Markov-Chebyshev inequality based on (30), which appears e.g. in [2, Section 2.4], shows that for any $\xi \in S^{n-1}$ and $t > 0$,

$$\nu\left(\{x \in \mathbb{R}^n \, ; \, |\langle \xi, x \rangle| \geq t E_\xi\}\right) \leq C \exp(-ct^\alpha). \tag{31}$$

Since $E_\xi \geq cR/\sqrt{n} \geq c$, we know that for any $\xi \in S^{n-1}$ and $t > 0$,

$$\gamma_n\left(\{x \in \mathbb{R}^n \, ; \, |\langle \xi, x \rangle| \geq t E_\xi\}\right) \leq \gamma_n\left(\{x \in \mathbb{R}^n \, ; \, |\langle \xi, x \rangle| \geq ct\}\right) \leq \tilde{C} e^{-\tilde{c}t^2} \leq \tilde{C} e^{-\hat{c}t^\alpha}. \tag{32}$$

Since $\mu = \gamma_n * \nu$, we deduce (29) from (31) and (32). □

Write g for the continuous density of the measure μ. Thus

$$g(x) = \frac{1}{N} \sum_{i=1}^N \frac{\varphi_n(x + R\Theta_i) + \varphi_n(x - R\Theta_i)}{2 \cdot (2\pi)^{n/2}} \qquad (x \in \mathbb{R}^n).$$

Lemma 4.6. *Assume that $n \geq \bar{C}$. Then with probability of at least $1 - n^{-n}$ the following holds: For any $\xi \in S^{n-1}$ and $t \geq 0$,*

$$\int_{t\xi + \xi^\perp} g \leq \frac{C}{n^{(2-\alpha)/4}},$$

where $C, \bar{C} > 0$ depend only on α.

Proof. We may assume that $\bar{C} \geq 5$ is large enough so that the assumption $n \geq \bar{C}$ implies that $10 n \log n \leq N \leq e^{10n}$. We may thus apply Proposition 3.4. According to the conclusion of this proposition, with probability of at least $1 - n^{-n}$ the following holds: For all $\xi \in S^{n-1}$ and $t \in \mathbb{R}$,

$$\frac{1}{N} \sum_{i=1}^N \varphi\left(t + R\xi \cdot \Theta_i\right) \leq \frac{Cn \log n}{\min\{N, n^3\}} + C \frac{\sqrt{n}}{R} \leq \frac{\tilde{C}}{n^{(2-\alpha)/4}}, \tag{33}$$

where we recall that $\tilde{C} = \tilde{C}(\alpha)$ depends solely on $\alpha \in (0, 2]$. Consequently, for all $\xi \in S^{n-1}$ and $t \geq 0$ we may use (21) and obtain

$$\int_{t\xi+\xi^\perp} g = \frac{1}{N} \sum_{i=1}^{N} \frac{\varphi\left(t + R\xi \cdot \Theta_i\right) + \varphi\left(R\xi \cdot \Theta_i - t\right)}{2 \cdot \sqrt{2\pi}} \leq \frac{C}{n^{(2-\alpha)/4}}. \qquad \square$$

It is well-known (see, e.g., [2, Section 2.4]) that if the probability measure μ admits ψ_α-tails with parameters (β, γ), then the following reverse Hölder inequality holds true: for any $\xi \in S^{n-1}$,

$$\sqrt{\int_{\mathbb{R}^n} |\langle x, \xi \rangle|^2 d\mu(x)} \leq C_{\beta,\gamma} \int_{\mathbb{R}^n} |\langle x, \xi \rangle| d\mu(x), \tag{34}$$

where $C_{\beta,\gamma} > 0$ depends only on β and γ.

Proof of Theorem 1.5. We may assume that $n \geq \bar{C}$, as otherwise the conclusion of the theorem is trivial. We may fix $\Theta_1, \ldots, \Theta_N \in S^{n-1}$ such that the events described in Lemma 4.3, Lemma 4.4, Lemma 4.5 and Lemma 4.6 hold true. Set

$$f(x) = g(x) \cdot 1_{3K}(x)/\mu(3K) \qquad (x \in T = 3K),$$

an even, continuous probability density supported on T. Note that $f \leq 2g$, according to Lemma 4.4. From Lemma 4.6, for any $\xi \in S^{n-1}$ and $t \geq 0$,

$$\int_{T\cap(\xi^\perp+t\xi)} f(y)dy \leq 2 \int_{\xi^\perp+t\xi} g(y)dy \leq \frac{C}{n^{(2-\alpha)/4}} \leq \frac{C}{n^{(2-\alpha)/4}} \cdot |T|^{-1/n},$$

where the last passage is the content of Lemma 4.3. This completes the proof of (i).

By Lemma 4.5, the probability measure μ admits ψ_α-tails with parameters (C, c). Write η for the probability measure whose density is f. According to Lemma 4.4 and the Cauchy-Schwartz inequality, for any $\xi \in S^{n-1}$,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\langle x, \xi \rangle| d\eta(x) &\geq \int_{\mathbb{R}^n} |\langle x, \xi \rangle| d\mu(x) - \int_{\mathbb{R}^n \setminus T} |\langle x, \xi \rangle| d\mu(x) \\
&\geq \int_{\mathbb{R}^n} |\langle x, \xi \rangle| d\mu(x) - \sqrt{\mu(\mathbb{R}^n \setminus T)} \sqrt{\int_{\mathbb{R}^n \setminus T} |\langle x, \xi \rangle|^2 d\mu(x)} \\
&\geq (1 - Ce^{-cn}) \int_{\mathbb{R}^n} |\langle x, \xi \rangle| d\mu(x), \qquad (35)
\end{aligned}$$

where we used (34) in the last passage. Since $\eta \leq 2\mu$, it thus follows from (35) that also η has ψ_α-tails with parameters $(\tilde{C}, \tilde{c})$. This completes the proof of (ii). $\square$

Appendix

In this appendix we indicate how Bourgain's argument may be modified in order to prove (4). We may normalize and assume that μ is a probability measure. The first observation is that denoting $M = \sup_{H \subseteq \mathbb{R}^n} \mu^+(K \cap H)$, we have that for all $\theta \in S^{n-1}$,

$$\begin{aligned}\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^2 d\mu(x) &= \int_{-\infty}^{\infty} t^2 \rho_\theta(t) dt = 4 \int_0^\infty t \left(\int_t^\infty \rho_\theta(s) ds \right) dt \\ &\geq 4 \int_0^{1/(2M)} t \left(\frac{1}{2} - \int_0^t \rho_\theta \right) dt \geq 4 \int_0^{1/(2M)} t \left(\frac{1}{2} - tM \right) dt = \frac{1}{12M^2},\end{aligned} \tag{36}$$

where $\rho_\theta(t) = \mu^+(K \cap H_{\theta,t})$ for $H_{\theta,t} = \{x \in \mathbb{R}^n \,;\, \langle x, \theta \rangle = t\}$. We will use inequality (36) in order to replace the hyperplane sections $\mu^+(K \cap H)$ in (4) by second moments of the probability measure μ. Thus, in order to prove (4), it suffices to show that

$$\inf_{\theta \in S^{n-1}} \sqrt{\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^2 d\mu(x)} \leq C(\beta, \gamma) \cdot n^{(2-\alpha)/4} \cdot \log n \cdot |K|^{1/n}. \tag{37}$$

The next step is to reduce matters to the case where μ is *isotropic*, in the sense that the integral $\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^2 d\mu(x)$ does not depend on $\theta \in S^{n-1}$. Indeed, there exists a volume-preserving linear transformation T such that the push-forward $T_*\mu$ is isotropic (see, e.g., [2, Section 2.3]). The replacement of μ by $T_*\mu$ and of K by $T(K)$ does not alter the right-hand side of (37), while it does not decrease the left-hand side. Hence from now on we assume that $L_\mu^2 := \int_{\mathbb{R}^n} |\langle x, \theta \rangle|^2 d\mu(x)$ does not depend on $\theta \in S^{n-1}$. In particular,

$$\int_{\mathbb{R}^n} |x|^2 d\mu(x) = \sum_{i=1}^n \int_{\mathbb{R}^n} |\langle x, e_i \rangle|^2 d\mu(x) = nL_\mu^2.$$

From the Markov-Chebyshev inequality it thus follows that $\mu(2\sqrt{n} L_\mu B^n) \geq 1/2$. As in the proof of Proposition 3.3.3 in [2], we may now use the ψ_α-condition in order to conclude that

$$\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^2 d\mu(x) \leq \tilde{C} \int_{2\sqrt{n} L_\mu B^n} |\langle x, \theta \rangle|^2 d\mu(x) \qquad \text{for all } \theta \in S^{n-1}, \tag{38}$$

where $\tilde{C}$, like all constants in this Appendix, depends solely on β and γ. The next step is "reduction to small diameter", which means that in proving (37) we would like to reduce matters to the case where μ is isotropic with

$$\mu(C\sqrt{n} L_\mu B^n) = 1, \tag{39}$$

for some constant C. The argument for this reduction in [2, Section 3.3.1], which involves conditioning μ to the ball $2\sqrt{n} L_\mu B^n$, applies almost verbatim thanks to (38). We may thus assume that (39) holds true, or equivalently that $|\langle x, \theta \rangle| \leq C L_\mu \sqrt{n}$ for all $\theta \in S^{n-1}$ and all $x \in \mathbb{R}^n$ in the support of the measure μ. Therefore,

$$\int_{\mathbb{R}^n} \exp\left\{ \left(\frac{\langle x, \theta \rangle}{\bar{C} L_\mu n^{(2-\alpha)/4}} \right)^2 \right\} d\mu(x) \leq \int_{\mathbb{R}^n} \exp\left\{ \frac{C^{2-\alpha}}{\bar{C}^2} \cdot \frac{|\langle x, \theta \rangle|^\alpha}{L_\mu^\alpha} \right\} d\mu(x) \leq \tilde{C}, \tag{40}$$

where the last inequality follows from the ψ_α-condition and a suitable choice of the constant $\bar{C}$ (see [2, Section 2.4] for standard computations related to the ψ_α-condition). Inequality (40) implies that for any $\theta \in S^{n-1}$,

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq C \cdot L_\mu \cdot n^{(2-\alpha)/4}. \tag{41}$$

Once we proved the ψ_2-estimate in (41), we may proceed as in the proof of Theorem 3.3.5 in [2], and use Talagrand's comparison theorem, the ℓ-position of Figiel and Tomczak-Jaegermann, and Pisier's estimate for the Rademacher projection. This establishes the desired inequality (37) in the isotropic case.

References

[1] Boucheron, S., Lugosi, G., Massart, P., *Concentration inequalities. A nonasymptotic theory of independence.* With a foreword by Michel Ledoux. Oxford University Press, Oxford, 2013.

[2] Brazitikos, S., Giannopoulos, A., Valettas, P., Vritsiou, B.-H., *Geometry of isotropic convex bodies.* Mathematical Surveys and Monographs, 196. American Mathematical Society, Providence, RI, 2014.

[3] J. Bourgain, *On high-dimensional maximal functions associated to convex bodies*, Amer. J. Math., 108, (1986), 1467-1476.

[4] J. Bourgain, *Geometry of Banach spaces and harmonic analysis* , Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, (1987), 871-878.

[5] J. Bourgain, *On the distribution of polynomials on high-dimensional convex sets*, Geom. aspects of Funct. Anal. (GAFA seminar notes), Israel Seminar, Springer Lect. Notes in Math. 1469 (1991), 127–137.

[6] G. Chasapis, A. Giannopoulos and D-M. Liakopoulos, *Estimates for measures of lower dimensional sections of convex bodies*, Adv. Math. 306 (2017), 880–904.

[7] A. Giannopoulos, *On some vector balancing problems,* Studia Math. 122 (1997), 225–234.

[8] E. D. Gluskin, *Extremal properties of orthogonal parallelepipeds and their applications to the geometry of Banach spaces,* Mat. Sb. (N.S.) 136(178) (1988), no. 1, 85–96; English translation in Math. USSR-Sb. 64 (1989), no. 1, 85-96.

[9] P. R. Goodey, E. Lutwak and W. Weil, *Functional analytic characterization of classes of convex bodies*, Math. Z. 222 (1996), 363–381.

[10] C. G. Khatri, *On certain inequalities for normal distributions and their applications to simultaneous confidence bounds,* Ann. Math. Statist. 38 (1967), 1853–1867.

[11] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. (GAFA) 16 (2006), 1274–1290.

[12] B. Klartag and E. Milman, *Centroid bodies and the logarithmic Laplace transform - a unified approach*, J. Funct. Anal. 262 (2012), 10–34.

[13] A. Koldobsky, *Intersection bodies, positive definite distributions and the Busemann-Petty problem*, Amer. J. Math. 120 (1998), 827–840.

[14] A. Koldobsky, *Fourier analysis in convex geometry*, Amer. Math. Soc., Providence RI, 2005.

[15] A. Koldobsky, *A hyperplane inequality for measures of convex bodies in* $\mathbb{R}^n, n \leq 4$, Discrete Comput. Geom. 47 (2012), 538–547.

[16] A. Koldobsky, *A* $\sqrt{n}$ *estimate for measures of hyperplane sections of convex bodies,* Adv. Math. 254 (2014), 33–40.

[17] A. Koldobsky, *Slicing inequalities for measures of convex bodies,* Adv. Math. 283 (2015), 473–488.

[18] A. Koldobsky and A. Pajor, *A remark on measures of sections of* L_p*-balls*, Geom. aspects of Funct. Anal. (GAFA seminar notes), Israel Seminar, Springer Lect. Notes in Math. 2169 (2017), 213–220.

[19] A. Koldobsky and A. Zvavitch, *An isomorphic version of the Busemann-Petty problem for arbitrary measures*, Geom. Dedicata 174 (2015), 261-277.

[20] G. Lozanovskii, *On some Banach lattices,* Siberian Math. J. 10 (1969), 419– 431.

[21] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. 71 (1988), 232–261.

[22] G. Pisier, *The volume of convex bodies and Banach space geometry*, Cambridge University Press, Cambridge, 1989.

[23] Z. Sidak, *On multivariate normal probabilities of rectangles: their dependence on correlations*, Ann. Math. Stat. 39 (1968), 1425-1434.

Bo'az Klartag: Department of Mathematics, Weizmann Institute of Science, Rehovot 76100 Israel, and School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978.

E-mail: `boaz.klartag@weizmann.ac.il`

Alexander Koldobsky: Department of Mathematics, University of Missouri, Columbia, MO 65211.

E-mail: `koldobskiya@missouri.edu`