Differential Geometry, homework assignment no. 1

Please submit your solution in pdf format by December 4 at 2PM at the link:
`https://www.dropbox.com/request/TlmYUm9nW42MBYUFfVh5`

You are asked to solve at least 8 out of the 9 questions and discuss your solution with Matan Eilat after submitting.

1. Let M and N be smooth manifolds. Introduce a natural smooth structure that makes the Cartesian product $M \times N$ a smooth manifold.

2. Let $f : M \to N$ be an immersion which is also a submersion. Recall the Inverse Function Theorem and explain why it is a local diffeomorphism, i.e., for any $p \in M$ there is an open neigborhoods U of $p \in M$ and V of $f(p) \in N$ such that $f|_U : U \to V$ is a diffeomorphism.

3. Let $f = (f_1, \ldots, f_k) : M \to \mathbb{R}^k$ be a submersion. Prove that at point $p \in M$, the differentials
$$(df_1)_p, \ldots, (df_k)_p : T_pM \to \mathbb{R} \, (= T_{f(p)}\mathbb{R}),$$
which are linear maps, are linearly independent (i.e., prove that there is no non-trivial linear combination of these linear maps that vanishes identically).

4. Write $\mathbb{RP}^n$ for the collection of lines through the origin in $\mathbb{R}^{n+1}$. Describe a natural smooth structure on $\mathbb{RP}^n$, and prove that it is a smooth n-dimensional manifold.

5. Let $\pi : S^n \to \mathbb{RP}^n$ be the map that associates with a point $x \in S^n \subset \mathbb{R}^{n+1}$ the line through the origin that passes through x. Prove that π is a smooth two-to-one map, which is a local diffeomorphism.

6. Let $f : N \to M$ be an embedding map. Prove that the smooth structure on $f(N)$ induced by the ambient manifold M coincides with the smooth structure on $f(N)$ that is induced from N through the homeomorphism f.

7. Let $w \in \mathbb{R}$. Suppose that M is a four-dimensional smooth manifold, and that the following Lorentz transformation is the transition map between two charts in M, representing two observers: for $(t, x, y, z) \in \mathbb{R}^4$,
$$T(t, x, y, z) = ((\cosh w)t - (\sinh w)x, (-\sinh w)t + (\cosh w)x, y, z).$$

In Special Relativity, this map represents the transition to the coordinate system of observer II, who passes through the origin at time 0, and moves at speed $v = \tanh w$ along the x-axis relative to observer I. Let the curve $\gamma(t) = (t, 1, 0, 0)$ for $t \in \mathbb{R}$ represent the world line of a particle at rest with respect to observer I. Describe how this world line appears to observer II, and explain the resulting effect of time dilation (in class we discussed length contraction).

8. Recall that our official definition of a smooth manifold requires that the topology is second countable (i.e., there exists a countable base for the topology). Prove that any smooth manifold is sigma-compact (i.e., it is covered by countably many compacts) and paracompact (i.e., every open cover of the manifold has a locally-finite refinement).

9. Hadamard's theorem: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map satisfying rank $f'(x) = n$ for all $x \in \mathbb{R}^n$ (i.e., a local diffeomorphism; it's both an immersion and a submersion). Assume that

$$\lim_{|x| \to \infty} |f(x)| = \infty$$

(equivalently, the preimage of a compact set is compact. This is known as a proper map). Prove that f is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Hint: If you know what a covering map is, you may use this. If not, think of the line segment between two points x and y with $f(x) = f(y)$. What is its image under f? What happens to the preimage when you slowly deform this image, without moving the point $p = f(x) = f(y)$?