Differential Geometry, homework assignment no. 2

Please submit your solution in pdf format by December 25 at 2PM at the link:
`https://www.dropbox.com/request/4kSsS5ZmGCl5QIPgDFrr`

You are asked to solve questions 1–5 and at least two out of questions 6–9.

1. Verify the Jacobi identity: For any vector fields X, Y and Z,
$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

2. (a) Let X be a vector field such that $Xf \equiv 0$ for any smooth, scalar function f. Prove that $X \equiv 0$.
   (b) Let f be a smooth, scalar function with $df \equiv 0$. Prove that f is locally-constant (i.e., constant on each connected component).

3. Recall the charts for the cotangent bundle T^*M we constructed in class (the "associated coordinates". In Lee's book these are "natural coordinates"). Show that they form a smooth atlas for T^*M. Bonus: Prove that T^*M is diffeomorphic to TM.

4. Let $\pi : \mathbb{R}^{2n} \setminus \{0\} \to \mathbb{RP}^{2n-1}$ be the map that associates with any point the line through the origin in which it lies. Prove that there is a 1-form ω on $\mathbb{RP}^{2n-1}$ with
$$\pi^*\omega = \sum_{i=1}^{n} \frac{x_{n+i}dx_i - x_i dx_{n+i}}{|x|^2}.$$

5. Let $f : \mathbb{C} \to \mathbb{C}$ be a smooth map. We write $Re(f), Im(f) : \mathbb{C} \to \mathbb{R}$ for the real and imaginary parts. Prove that f is a local diffeo if and only if $dRe(f) \wedge dIm(f) \neq 0$, and that this condition is equivalent to $df \wedge d\bar{f} \neq 0$.

6. Let M be a smooth manifold, $p \in M$, and let $X_p : C^\infty(M) \to \mathbb{R}$ be a linear map satisfying the Leibniz rule:
$$X_p(fg) = f(p)X_p(g) + g(p)X_p(f) \qquad \text{for all } f, g \in C^\infty(M).$$
Prove that there exists a tangent vector $v \in T_pM$ such that $X_pf = v(f)$ for all $f \in C^\infty(M)$. [Hint: Perhaps show that if $f = c + gh$ for some constant c and functions $g, h \in C^\infty(M)$ vanishing at p, then $X_p(f) = 0$. Conclude that $X_p(f) = 0$ when f vanishes to first order at p]

7. Let $X_1, \ldots, X_n$ be smooth vector fields in an n-dimensional manifold M with $[X_i, X_j] \equiv 0$ for all i, j. Assume that they are linearly independent at a given point $p \in M$. Define

$$f(t_1, \ldots, t_n) = \varphi_{t_1}^{X_1} \circ \varphi_{t_2}^{X_2} \ldots \varphi_{t_n}^{X_n}(p),$$

where $(\varphi_t^{X_i})_{t \in \mathbb{R}}$ is the one-paramater group of transformations associated with X_i. Prove that f is a local diffeomorphism, from a neigborhood of the origin in $\mathbb{R}^n$ to a neighborhood of p in M, with

$$f_* \left(\frac{\partial}{\partial t_i} \right) = X_i.$$

[You may assume without proof that f is smooth]

8. Let M be a manifold, and let V be a smooth vector field on M. Consider the transformations $(\varphi_t)_{t \in \mathbb{R}}$ associated with V. Prove that when M is compact, $\varphi_t : M \to M$ is a diffeomorphism for any $t \in \mathbb{R}$. [hint: Prove first that for some $\varepsilon > 0$, $\varphi_t(x)$ is well-defined for all $x \in M$ and $|t| < \varepsilon$].

9. Let C be a connected, one-dimensional manifold. Prove that C is diffeomorphic to S^1 if it is compact, and otherwise it is diffeomorphic to $(0, 1)$.
Bonus: Read in Lee's book or elsewhere the definition of a manifold with boundary. Prove that if we allow boundary, then there are two additional options for C: It can also be diffeomorphic to $[0, 1]$ or to $[0, 1)$.