Differential Geometry, homework assignment no. 4

Please submit your solution in pdf format by February 26 at 2PM at the link:
`https://www.dropbox.com/request/Ya8WOVCKi1082WI9yZfo`

You are asked to solve questions 1–4 and at least three out of questions 5–10.

1. A top form ω on a Riemannian manifold M is called a Riemannian volume form if

$$\omega(e_1, \ldots, e_n) = \pm 1$$

for any orthonormal basis $e_1, \ldots, e_n \in T_pM$. Let $M \subset \mathbb{R}^n$ be a hypersurface equipped with the induced Riemannian structure, with unit normal N. Prove that $i_N\omega$ is a Riemannian volume form on M, where $\omega = dx_1 \wedge \ldots \wedge dx_n$ is the standard volume form in $\mathbb{R}^n$.

2. We work in $\mathbb{R}^4$ with coordinates t, x_1, x_2, x_3. Suppose that

$$F = \sum_{i=1}^{3} E_i \, dx_i \wedge dt + \sum_{i=1}^{3} B_i \, dx_{i+1} \wedge dx_{i+2},$$

where $x_{i+3} = x_i$ and E_i, B_i are scalar functions on $\mathbb{R}^4$. Verify that $dF = 0$ if and only the two homogeneous Maxwell equations (in certain units) hold:

$$\begin{aligned} \text{div}(B) &= 0, \\ \text{curl}(E) + \frac{\partial B}{\partial t} &= 0, \end{aligned}$$

where div and curl act in the x-variables. Define the Hodge $*$ operator as a linear operator on 2-forms via

$$\begin{aligned} *(dx_{i+1} \wedge dx_{i+2}) &= dt \wedge dx_i, \\ *(dx_i \wedge dt) &= dx_{i+1} \wedge dx_{i+2}. \end{aligned}$$

Set $j = d(*F)$. Identify the components J_1, J_2, J_3, ρ of the 3-form j in $\mathbb{R}^4$ and prove

$$\begin{aligned} \text{div}(E) &= \rho, \\ \text{curl}(B) - \frac{\partial E}{\partial t} &= J. \end{aligned}$$

3. Use Stokes' formula to prove *Cauchy's integral formula*: For any bounded, open set $\Omega \subset \mathbb{C}$ with a smooth boundary and for any smooth function $f : \Omega \to \mathbb{C}$,
$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0} dz - \frac{1}{\pi} \int_{\Omega} \frac{\partial f / \partial \bar{z}}{z - z_0} dx \wedge dy \quad (z_0 \in \Omega).$$
Hint: Consider a small disc around z_0 whose radius ε will later tend to zero.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth map with $k < n$ and set $M = \{x \in \mathbb{R}^n \mid f(x) = 0\}$. Assume that $M \neq \emptyset$ and that zero is a regular value of f (i.e., that the matrix $f'(x) \in \mathbb{R}^{n \times k}$ is of rank k for all $x \in M$). Prove that M is orientable.

5. Recall the n-torus $\mathbb{T}^n = (S^1)^n$. Prove that
$$\dim H^k(\mathbb{T}^n) \geq \binom{n}{k}. \tag{1}$$
Hint: Look at integrals on that many sub-tori $(S^1)^k$. Bonus: Show that equality holds in equation (1).

6. Let X be a vector field on a manifold M and write $(\varphi_t)_{t \in \mathbb{R}}$ for the associated one-parameter group of transformations. Let ω be a k-form, and consider the Lie derivative
$$L_X \omega := \left. \frac{d}{dt} \varphi_t^* \omega \right|_{t=0},$$
which is a k-form on M. Prove Cartan's formula:
$$L_X \omega = i_X(d\omega) + d(i_X \omega).$$
Hint: Maybe use the case of 1-forms, and think about the effect of wedge products.

7. De Rham cohomology with compact support: Let ω be a compactly-supported top form on $\mathbb{R}^n$ with
$$\int_{\mathbb{R}^n} \omega = 0.$$
Prove that there exists a compactly-supported $(n-1)$-form α with $\omega = d\alpha$.

*8. Prove that a smooth, odd function $f : S^n \to S^n$ has an odd degree. *Hint:* Maybe induction on the dimension.

9. Use the conclusion of the previous question in order to prove the *Borsuk-Ulam theorem*: Any smooth, odd map $f : S^n \to \mathbb{R}^n$ has to vanish somewhere in the sphere.

   Hint: Otherwise, obtain a smooth odd map $f : S^n \to S^{n-1}$. Identify S^{n-1} with the equator in S^n, and restrict f to the upper hemi-sphere to obtain a smooth, odd map $F : \overline{B^n} \to S^{n-1}$ with $F(S^{n-1}) \subseteq S^{n-1}$.

10. Let $\Omega \subset \mathbb{C}$ be a bounded, open set and let $f : \Omega \to \mathbb{C}$ be a smooth function. Denote $M = \{(w, z) \in \mathbb{C}^2 \mid z \in \Omega, w = f(z)\}$. Prove Wirtinger's inequality:
$$\frac{i}{2} \int_M (dz \wedge d\bar{z} + dw \wedge d\bar{w}) \leq \text{Area}(M)$$
with equality if and only if f is holomorphic.