Logarithmic bounds for isoperimetry and slices of convex sets

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Abstract. *We prove that the Bourgain slicing conjecture and the Kannan-Lovász-Simonovits (KLS) isoperimetric conjecture in $\mathbb{R}^n$ hold true up to a factor of $\sqrt{\log n}$. A new ingredient used in the proof is an improved log-concave Lichnerowicz inequality.*

Keywords. Isoperimetric inequalities, hyperplane sections, high-dimensional convex sets.

1. INTRODUCTION

Bourgain's slicing problem asks whether any convex body $K \subseteq \mathbb{R}^n$ of volume one admits a hyperplane $H \subseteq \mathbb{R}^n$ such that

$$Vol_{n-1}(K \cap H) > c,$$

for a universal constant $c > 0$. Here Vol_{n-1} stands for $(n-1)$-dimensional volume and a hyperplane is a one-codimensional affine subspace. We refer the reader to Klartag and Milman [35] and references therein for background on this problem, its equivalent formulations and its applications. For $n \geq 2$ define

$$\frac{1}{L_n} = \inf_{K \subseteq \mathbb{R}^n} \sup_{H \subseteq \mathbb{R}^n} Vol_{n-1}(K \cap H),$$

where the infimum runs over all convex bodies $K \subseteq \mathbb{R}^n$ of volume one, and the supremum runs over all hyperplanes $H \subseteq \mathbb{R}^n$. Thus, a convex body of volume one in $\mathbb{R}^n$ has a hyperplane section whose $(n-1)$-dimensional volume is at least $1/L_n$, and Bourgain's slicing problem asks whether $L_n < C$ for a universal constant $C > 0$.

For decades the best known bounds have been $L_n \leqslant C n^{1/4} \log n$ proven in Bourgain [7, 8] and $L_n \leqslant C n^{1/4}$ proven in [29]. Two years ago, a breakthrough by Chen [14] led to the bound $L_n \leqslant \exp\left(C\sqrt{\log n \cdot \log\log n}\right)$ which was subsequently improved to $L_n \leqslant C \log^4 n$ by Klartag and Lehec [34], to $L_n \leqslant C \log^{2.223\ldots} n$ by Jambulapati, Lee and Vempala [25] and to $L_n \leqslant C \log^{2.082\ldots}$ by Lehec (personal communication). In this paper we obtain a further improvement:

Theorem 1.1. *For $n \geqslant 2$,*

$$L_n \leqslant C\sqrt{\log n},$$

where $C > 0$ is a universal constant.

A probability density $\rho : \mathbb{R}^n \to [0, \infty)$ is *log-concave* if its support $\{x \in \mathbb{R}^n \, ; \rho(x) > 0\}$ is a convex set and $\log \rho$ is concave in the support of ρ. A random vector X in $\mathbb{R}^n$ is log-concave if it is supported in an affine subspace and has a log-concave density in this subspace. The uniform distribution on a convex body is log-concave, as well as the Gaussian distributions. A random vector $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ is *isotropic* if it has finite second moments and for $i, j = 1, \ldots, n$,

$$\mathbb{E}X_i = 0 \qquad \text{and} \qquad \mathbb{E}X_i X_j = \delta_{ij}$$

where δ_{ij} is Kronecker's delta. When we say that a Borel probability measure μ on $\mathbb{R}^n$ is log-concave or is isotropic, we mean that the random vector X with law μ has the corresponding property. When μ is isotropic and has a log-concave density ρ, for any hyperplane $H \subseteq \mathbb{R}^n$ passing through the origin,

$$(1.1) \qquad \frac{1}{\sqrt{12}} \leqslant \int_H \rho \leqslant \frac{1}{\sqrt{2}}.$$

This is proven in Hensley [24] and Fradelizi [22], using the Prékopa-Leindler inequality and one-dimensional analysis. Moreover, if $H^+ \subseteq \mathbb{R}^n$ is a half-space whose boundary is H then Grünbaum's theorem [23] states that

$$(1.2) \qquad \frac{1}{e} \leqslant \mu(H^+) \leqslant 1 - \frac{1}{e}.$$

Thus the boundary measure of any such half-space has the same order of magnitude as the measure of the half-space itself. In fact, it suffices to assume that μ is centered for (1.2) to hold true, with no need for isotropicity.

The *Poincaré constant* of a random vector X in $\mathbb{R}^n$, denoted by $C_P(X)$, is the infimum over all $C \geqslant 0$ such that for any locally-Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying $\mathbb{E}|\nabla f(X)|^2 < \infty$,

$$(1.3) \qquad Var(f(X)) \leqslant C \cdot \mathbb{E}|\nabla f(X)|^2.$$

Given a probability measure μ in $\mathbb{R}^n$ we write $C_P(\mu) = C_P(X)$ where X is the random vector with law μ. When μ has a log-concave density ρ, its *isoperimetric constant* is

$$\frac{1}{\psi_\mu} = \inf_{A \subseteq \mathbb{R}^n} \left\{ \frac{\int_{\partial A} \rho}{\min\{\mu(A), 1 - \mu(A)\}} \right\}$$

where the infimum runs over all open sets $A \subseteq \mathbb{R}^n$ with smooth boundary satisfying $0 < \mu(A) < 1$. This isoperimetric constant is finite and positive, as proven in Bobkov [5]. The inequalities of Cheeger [13] and Buser and Ledoux [11, 37] imply that the Poincaré constant and the isoperimetric constant are closely related. Namely,

$$(1.4) \qquad \frac{1}{4} \leqslant \frac{\psi_\mu^2}{C_P(\mu)} \leqslant \pi.$$

The value of the numerical constant on the right-hand side of (1.4) was found by De Ponti and Mondino [18] using the technique from Ledoux [37]. We define

$$(1.5) \qquad \psi_n = \sup_\mu \psi_\mu$$

where the supremum runs over all isotropic, log-concave probability measures in $\mathbb{R}^n$. The Kannan-Lovász-Simonovits (KLS) conjecture [28] suggests that $\psi_n < C$ for a universal constant $C > 0$. If correct then it would imply that the most efficient way to partition a convex body into two pieces of equal mass so as to minimize their interface is a hyperplane bisection, up to a universal constant.

The fact that the KLS conjecture implies Bourgain's slicing conjecture was announced by Ball in 2003; the proof, using the heat equation and yielding the estimate $L_n \leqslant \exp(C\psi_n^2)$, was published in the work of Ball and Nguyen [3]. In the meantime, Eldan and Klartag [20] used the logarithmic Laplace transform and proved the bound

$$(1.6) \qquad L_n \leqslant C\psi_n,$$

where $C > 0$ is a universal constant. Substantial progress towards the KLS conjecture started roughly a decade ago, with Eldan's stochastic localization method [19]. By utilizing this method, Lee and Vempala [26] proved the bound $\psi_n \leqslant Cn^{1/4}$, that was improved to $\psi_n \leqslant \exp(C\sqrt{\log n \cdot \log\log n})$ by Chen [14], to $\psi_n \leqslant C\log^5 n$ by Klartag and Lehec [34], to $\psi_n \leqslant C\log^{3.223\cdots} n$ by Jambulapati, Lee and Vempala [25] and to $\psi_n \leqslant C\log^{3.082\cdots}$ by Lehec (personal communication). Theorem 1.1 follows from (1.6) and the following:

Theorem 1.2. *For $n \geqslant 2$,*

$$\psi_n \leqslant C\sqrt{\log n},$$

where $C > 0$ is a universal constant.

Already in the case of unconditional convex bodies, Theorem 1.2 improves upon the previously known bound. That bound was obtained in [30].

Our new ingredient is an improved log-concave Lichnerowicz inequality. We say that a probability measure μ in $\mathbb{R}^n$ is *t-uniformly log-concave*, for $t \geqslant 0$, if the probability measure ν whose density with respect to μ is proportional to $e^{t|x|^2/2}$ exists and is log-concave.

The *log-concave Lichnerowicz inequality* states that for any $t > 0$ and a t-uniformly log-concave probability measure μ in $\mathbb{R}^n$,

(1.7)
$$C_P(\mu) \leqslant 1/t.$$

The name of this inequality stems from an analogy to investigations by Lichnerowicz in Riemannian geometry, see [2, 36]. Inequality (1.7) also follows from the Brascamp-Lieb inequality [9]. Write $Cov(\mu) = (Cov_{ij}(\mu))_{i,j=1,\ldots,n} \in \mathbb{R}^{n \times n}$ for the covariance matrix of the log-concave probability measure μ, defined via

$$Cov_{ij}(\mu) = \int_{\mathbb{R}^n} x_i x_j \, d\mu(x) - \int_{\mathbb{R}^n} x_i \, d\mu(x) \int_{\mathbb{R}^n} x_j \, d\mu(x).$$

The covariance matrix is well-defined, since a log-concave probability density decays exponentially at infinity (e.g. [10, Lemma 2.2.1]). The covariance matrix is a symmetric, positive semi-definite matrix, and its operator norm is denoted by $\|Cov(\mu)\|_{op}$.

Theorem 1.3 ("improved log-concave Lichnerowicz inequality"). *Let $t > 0$ and let μ be a t-uniformly log-concave probability measure in $\mathbb{R}^n$. Then,*

(1.8)
$$C_P(\mu) \leqslant \sqrt{\frac{\|Cov(\mu)\|_{op}}{t}} \leqslant \frac{1}{t}.$$

The bound in (1.8) is the geometric average of the Lichnerowicz bound (1.7) with the KLS conjectural bound $C_P(\mu) \lesssim \|Cov(\mu)\|_{op}$, where $A \lesssim B$ is a shorthand for $A \leqslant C \cdot B$ where $C > 0$ is a universal constant. Theorem 1.3 is proven in Section 2 by using the Bochner formula in order to analyze the first non-trivial eigenfunction of the Laplacian associated with μ. The proof of Theorem 1.2 is given in Section 3, and it combines Theorem 1.3 with some properties of Eldan's stochastic localization.

Our usage of stochastic localization is relatively "soft" compared to recent works in this field. Stochastic processes are crucial in this proof only as a tool for bounding the covariance process up until a certain time determined by ψ_n, see the bound (3.13) below. It could be interesting to try and replace the role of stochastic processes in this argument with differentiations along the heat flow, integrations by parts, and 3-tensor analysis.

We write $\nabla^2 u$ for the Hessian matrix of the function $u : \mathbb{R}^n \to \mathbb{R}$. The scalar product of $x, y \in \mathbb{R}^n$ is $x \cdot y = \langle x, y \rangle = \sum_i x_i y_i$. The Euclidean norm of $x \in \mathbb{R}^n$ is $|x| = \sqrt{x \cdot x}$, and $S^{n-1} = \{x \in \mathbb{R}^n ; |x| = 1\}$ is the unit sphere centered at the origin. A smooth function is C^∞-smooth, and log stands for the natural logarithm. We write $C, c, \tilde{C}, \tilde{c}, \bar{C}$ etc. to denote various positive universal constants whose value may change from one line to the next.

2. EIGENFUNCTIONS WITH A PREFERRED DIRECTION

Let μ be a probability measure in $\mathbb{R}^n$ with a log-concave probability density ρ. We say that μ is *regular* if its density ρ is smooth and positive in $\mathbb{R}^n$ and the following two requirements hold:

(i) There exists $\varepsilon > 0$ such that for all $x \in \mathbb{R}^n$, denoting $\psi = -\log \rho$,

$$(2.1) \qquad \varepsilon \cdot \mathrm{Id} \leqslant \nabla^2 \psi(x) \leqslant \frac{1}{\varepsilon} \cdot \mathrm{Id}$$

in the sense of symmetric matrices.

(ii) The function ψ, as well as each of its partial derivatives, grows at most polynomially at infinity.

Lemma 2.1. *Let μ be an absolutely-continuous, log-concave probability measure in $\mathbb{R}^n$ and let $0 < \varepsilon < t$. Then there exists a regular, log-concave probability measure ν in $\mathbb{R}^n$ such that*

$$C_P(\nu) \geqslant C_P(\mu) - \varepsilon \qquad and \qquad \|Cov(\mu) - Cov(\nu)\|_{op} < \varepsilon.$$

Moreover, if μ is t-uniformly log-concave, then ν is $(t - \varepsilon)$-uniformly log-concave.

The proof of this approximation lemma is deferred to the Appendix below. Assume from now on that μ is a probability measure on $\mathbb{R}^n$ with a regular, log-concave probability density $\rho = e^{-\psi}$. As in Klartag and Putterman [33], we say that a function $f : \mathbb{R}^n \to \mathbb{R}$ has subexponential decay relative to ρ if there exist $C, a > 0$ such that

$$(2.2) \qquad |f(x)| \leqslant \frac{C}{\sqrt{\rho(x)}} e^{-a|x|} \qquad\qquad (x \in \mathbb{R}^n).$$

We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *μ-tempered* if it is smooth and if all of its partial derivatives of all orders have subexponential decay relative to ρ. A log-concave probability density decays exponentially at infinity. Therefore if a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ grows at most polynomially at infinity, as do all of its partial derivatives, then it is μ-tempered. A μ-tempered function is clearly in $L^2(\mu)$. Write $\mathscr{F}_\mu$ for the collection of all μ-tempered functions on $\mathbb{R}^n$. The Laplace operator associated with μ is defined for $u \in \mathscr{F}_\mu$ via

$$(2.3) \qquad Lu = L_\mu u = \Delta u - \nabla \psi \cdot \nabla u.$$

For any $u, v \in \mathscr{F}_\mu$ we have the integration by parts formula

$$(2.4) \qquad \int_{\mathbb{R}^n} (Lu) v \, d\mu = -\int_{\mathbb{R}^n} \langle \nabla u, \nabla v \rangle \, d\mu$$

and the (integrated) Bochner formula

$$(2.5) \qquad \int_{\mathbb{R}^n} (Lu)^2 \, d\mu = \int_{\mathbb{R}^n} \|\nabla^2 u\|_{HS}^2 \, d\mu + \int_{\mathbb{R}^n} \langle (\nabla^2 \psi) \nabla u, \nabla u \rangle \, d\mu,$$

where $\|\nabla^2 u\|_{HS}$ is the Hilbert-Schmidt norm of the Hessian matrix $\nabla^2 u$. Formulae (2.4) and (2.5) are proven by intergation by parts, see e.g., Ledoux [36, Section 2.3]. The regularity of ρ and the μ-temperedness of u, v are used in order to discard the boundary terms. The integrated Bochner formula (2.5) is related to the commutation relation $L(\nabla u) = \nabla Lu + (\nabla^2 \psi) \nabla u$, and is reminiscent of similar formulae in Riemannian geometry.

The operator $-L$ is essentially self-adjoint and positive semi-definite in $\mathscr{F}_\mu \subseteq L^2(\mu)$ with a discrete spectrum (see e.g., [2, Corollary 4.10.9] and [33, Proposition A.1]).

Lemma 2.2. *All eigenfunctions of L_μ are μ-tempered.*

The proof of Lemma 2.2 is discussed in the Appendix; as in [33, Section 2] this proof is reduced to known results on exponential decay of eigenfunctions of Schrödinger operators. The minimal eigenvalue of $-L$ is 0, which is a simple eigenvalue corresponding to a constant eigenfunction. We write

$$\lambda = \lambda(\mu) > 0$$

for the minimal non-zero eigenvalue of $-L$. The positive semi-definite operator $-L$ has no spectrum in the interval $(0, \lambda)$, while 0 is an eigenvalue of multiplicity one. It follows that for any $f \in \mathscr{F}_\mu$ with $\int f d\mu = 0$,

$$(2.6) \qquad \lambda \int_{\mathbb{R}^n} f^2 d\mu \le \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$$

with equality if and only if f is a eigenfunction of L corresponding to the first non-zero eigenvalue. Inequality (2.6) is sometimes referred to as the Poincaré inequality for the measure μ. Indeed, the space of compactly-supported, smooth functions is dense in the Sobolev space $H^1(\mu)$, see the Appendix of [4], and hence

$$\lambda(\mu) = 1/C_P(\mu).$$

One way to prove the log-concave Lichnerowicz inequality (1.7) is to substitute the eigenfunction f into the Bochner formula (2.5) and obtain the inequality $\lambda^2 \ge t\lambda$ which is equivalent to (1.7). The following is a quantitative, log-concave version of Corollary 1 from [30].

Proposition 2.3. *Let $f \in \mathscr{F}_\mu$ be an eigenfunction of $-L$ corresponding to the eigenvalue $\lambda = \lambda(\mu)$ and normalized so that $\|f\|_{L^2(\mu)} = 1$. Then,*

$$(2.7) \qquad \left| \int_{\mathbb{R}^n} \nabla f d\mu \right|^2 \ge \frac{1}{\lambda} \int_{\mathbb{R}^n} [(\nabla^2 \psi) \nabla f \cdot \nabla f] d\mu,$$

and

$$(2.8) \qquad \left| \int_{\mathbb{R}^n} \nabla f d\mu \right|^2 = \lambda^2 \left| \int_{\mathbb{R}^n} f(x) x d\mu(x) \right|^2 \le \lambda^2 \|Cov(\mu)\|_{op}.$$

Proof. By the Bochner formula and the Poincaré inequality for $\partial^i f$ ($i = 1, \ldots, n$),

$$\lambda^2 = \int_{\mathbb{R}^n} (Lf)^2 d\mu = \int_{\mathbb{R}^n} [(\nabla^2 \psi) \nabla f \cdot \nabla f] d\mu + \int_{\mathbb{R}^n} \|\nabla^2 f\|_{HS}^2 d\mu$$

$$\ge \int_{\mathbb{R}^n} [(\nabla^2 \psi) \nabla f \cdot \nabla f] d\mu + \lambda \left[\int_{\mathbb{R}^n} |\nabla f|^2 d\mu - \left| \int_{\mathbb{R}^n} \nabla f d\mu \right|^2 \right]$$

$$(2.9) \qquad = \int_{\mathbb{R}^n} [(\nabla^2 \psi) \nabla f \cdot \nabla f] d\mu + \lambda^2 - \lambda \left| \int_{\mathbb{R}^n} \nabla f d\mu \right|^2.$$

This implies (2.7). As for (2.8), note that for any $\theta \in S^{n-1}$, denoting $g(x) = \langle x, \theta \rangle$,

$$(2.10) \quad \int_{\mathbb{R}^n} \langle \nabla f, \theta \rangle d\mu = \int_{\mathbb{R}^n} \langle \nabla f(x), \nabla g(x) \rangle d\mu(x) = -\int_{\mathbb{R}^n} (Lf)g \, d\mu = \lambda \int_{\mathbb{R}^n} f(x) \langle x, \theta \rangle d\mu(x).$$

Denote $E = \int_{\mathbb{R}^n} \langle x, \theta \rangle d\mu(x)$. We use the facts that $\int f d\mu = 0$, $\int f^2 d\mu = 1$ and the Cauchy-Schwartz inequality, and obtain

$$\left| \int_{\mathbb{R}^n} \langle \nabla f, \theta \rangle d\mu \right|^2 = \lambda^2 \left| \int_{\mathbb{R}^n} f(x)(\langle x, \theta \rangle - E) d\mu(x) \right|^2 \leqslant \lambda^2 \int_{\mathbb{R}^n} (\langle x, \theta \rangle - E)^2 d\mu(x).$$

This implies (2.8), since the last integral equals $\langle Cov(\mu)\theta, \theta \rangle$. $\qquad\qquad \square$

We intuitively think of a function $f \in \mathscr{F}_\mu$ as having a "preferred direction" when

$$\left| \int_{\mathbb{R}^n} \nabla f d\mu \right|^2 \gtrsim \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.$$

The property that the eigenfunction f from Proposition 2.3 has a preferred direction is in the spirit of the "hot spots conjecture" [27], as we learned from David Jerison. The KLS conjecture suggests that

$$\lambda(\mu) \cdot \|Cov(\mu)\|_{op} \geqslant c$$

for a universal constant $c > 0$. Since $\int |\nabla f|^2 d\mu = \lambda$ for the eigenfunction f from Proposition 2.3, we see from (2.8) that if the eigenfunction has a preferred direction, then the conclusion of the KLS conjecture holds.

Proof of Theorem 1.3. We may assume that μ is not supported in an affine subspace $E \subsetneq \mathbb{R}^n$ (otherwise, we work in this subspace). Since μ is log-concave, this means that the measure μ is necessarily absolutely-continuous in $\mathbb{R}^n$. An approximation argument based on Lemma 2.1 shows that it suffices to prove the theorem for an absolutely-continuous, regular, log-concave probability measure μ on $\mathbb{R}^n$.

Let f be an eigenfunction of the operator $-L_\mu$ corresponding to the eigenvalue $\lambda = \lambda(\mu)$ and normalized so that $\|f\|_{L^2(\mu)} = 1$. By combining (2.7) and (2.8) we obtain

$$(2.11) \qquad\qquad \int_{\mathbb{R}^n} \left[(\nabla^2 \psi) \nabla f \cdot \nabla f \right] d\mu \leqslant \lambda^3 \cdot \|Cov(\mu)\|_{op}.$$

Since μ is t-uniformly log-concave, we know that $\nabla^2 \psi(x) \geqslant t$ for all $x \in \mathbb{R}^n$. Hence the left-hand side of (2.11) is at least

$$(2.12) \qquad\qquad t \int_{\mathbb{R}^n} |\nabla f|^2 d\mu = t\lambda.$$

Hence (2.11) implies that $\lambda^2 \|Cov(\mu)\|_{op} \geqslant t$. This proves the left-hand side inequality in (1.8). For the right-hand side inequality, for any $\theta \in S^{n-1}$ we set $g_\theta(x) = \langle x - b_\mu, \theta \rangle$ where

$$b_\mu = \int_{\mathbb{R}^n} x \, d\mu(x) \in \mathbb{R}^n$$

is the barycenter of μ. We now use the Poincaré inequality (2.6) as follows:

$$\|Cov(\mu)\|_{op} = \sup_{\theta \in S^{n-1}} Cov(\mu)\theta \cdot \theta = \sup_{\theta \in S^{n-1}} \int_{\mathbb{R}^n} g_\theta^2 d\mu \leq \frac{1}{\lambda} \sup_{\theta \in S^{n-1}} \int_{\mathbb{R}^n} |\nabla g_\theta|^2 d\mu = \frac{1}{\lambda} \leq t,$$

where we used the log-concave Lichnerowicz inequality (1.7) in the last passage. $\qquad\square$

3. LOGARITHMIC BOUND FOR THE KLS CONSTANT

We use the notation from [33, Section 4]. Let μ be a probability measure on $\mathbb{R}^n$ with a regular, log-concave probability density $\rho = e^{-\psi}$. For $t \geq 0$ and $\theta \in \mathbb{R}^n$ we denote

$$(3.1) \qquad\qquad p_{t,\theta}(x) = \frac{1}{Z(t,\theta)} e^{\langle \theta, x \rangle - t|x|^2/2} \rho(x) \qquad\qquad (x \in \mathbb{R}^n)$$

where $Z(t,\theta) = \int_{\mathbb{R}^n} e^{\langle \theta, x \rangle - t|x|^2/2} \rho(x) dx$. The barycenter and covariance matrix of the probability density $p_{t,\theta}$ are denoted by

$$a(t,\theta) = \int_{\mathbb{R}^n} x p_{t,\theta}(x)\, dx \in \mathbb{R}^n$$

and

$$A(t,\theta) = \int_{\mathbb{R}^n} (x \otimes x) p_{t,\theta}(x)\, dx - a(t,\theta) \otimes a(t,\theta) \in \mathbb{R}^{n \times n},$$

where $x \otimes x = (x_i x_j)_{i,j=1,\ldots,n} \in \mathbb{R}^{n \times n}$. Write $\mu_{t,\theta}$ for the probability measure whose density is the regular, log-concave probability density $p_{t,\theta}$. We abbreviate

$$\lambda(t,\theta) = \lambda(\mu_{t,\theta}),$$

the first non-zero eigenvalue of the operator $-L_{\mu_{t,\theta}}$. Since ρ is log-concave, the probability measure $\mu_{t,\theta}$ is t-uniformly log-concave, as we see from formula (3.1). We may therefore apply the improved log-concave Lichnerowicz inequality, which is Theorem 1.3 above, and obtain the bound

$$(3.2) \qquad\qquad \lambda(t,\theta) \geq \sqrt{\frac{t}{\|A(t,\theta)\|_{op}}} \geq t.$$

Given $f \in L^1(\mu)$ we write

$$M_f(t,\theta) = \int_{\mathbb{R}^n} f(x) p_{t,\theta}(x) dx.$$

This is a smooth function of $\theta \in \mathbb{R}^n$, and by differentiating under the integral sign we obtain

$$\nabla_\theta M_f(t,\theta) = \int_{\mathbb{R}^n} (x - a(t,\theta)) f(x) p_{t,\theta}(x) dx$$

$$(3.3) \qquad\qquad = \int_{\mathbb{R}^n} (x - a(t,\theta))(f(x) - M_f(t,\theta)) p_{t,\theta}(x) dx.$$

The "tilt process" is the stochastic process $(\theta_t)_{t \geq 0}$ attaining values in $\mathbb{R}^n$ and defined via the stochastic differential equation

$$(3.4) \qquad\qquad \theta_0 = 0, \quad d\theta_t = dW_t + a(t,\theta_t) dt,$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion in $\mathbb{R}^n$ with $W_0 = 0$. The existence and unique-ness of a strong solution to (3.4) are proven via a standard argument (see, e.g. Chen [14]). As explained in [33, Section 4], the process $(\theta_t)_{t \geq 0}$ coincides in law with the pro-cess $(tX + W_t)_{t \geq 0}$ where X is a random vector with law μ, independent of the Brownian motion $(W_t)_{t \geq 0}$. Setting $p_t(x) = p_{t,\theta_t}(x)$ and $a_t = a(t, \theta_t)$ we obtain from the Itô formula that

$$dp_t(x) = p_t(x) \langle x - a_t, dW_t \rangle \qquad (x \in \mathbb{R}^n)$$

with $p_0(x) = \rho(x)$. This stochastic process is central in the theory of non-linear filtering, see e.g. Chiganski [15, Chapter 6] and references therein for its analysis and history. Its usefulness for proving isoperimetric inequalities and functional inequalities, for bound-ing mixing times, and for analyzing probability measures with convexity properties was realized by Eldan [19], Lee and Vempala [26], Chen [14] and others. The process $(p_t(x))_{t \geq 0}$ is a martingale with respect to the filtration induced by the Brownian motion, and in par-ticular

(3.5) $$\mathbb{E} p_t(x) = p_0(x) = \rho(x) \qquad (t \geq 0, x \in \mathbb{R}^n).$$

Abbreviate $\lambda_t = \lambda(t, \theta_t)$, $A_t = A(t, \theta_t)$ and note that $\lambda_0 = \lambda(\mu)$.

Lemma 3.1. *For any $t > 0$ and $f \in L^2(\mu)$,*

$$\mathbb{E} Var_{p_t}(f) \leq Var_{p_0}(f) \leq \left(2 + \frac{t}{\lambda_0}\right) \mathbb{E} Var_{p_t}(f)$$

where we write $Var_{p_t}(f) = \int_{\mathbb{R}^n} f^2 p_t - (\int_{\mathbb{R}^n} f p_t)^2$.

Proof. Set $M_t = M_f(t, \theta_t) = \int f p_t$. It follows from (3.5) and Fubini's theorem that

(3.6) $$Var_{p_0}(f) = \int_{\mathbb{R}^n} (f - M_0)^2 p_0 = \mathbb{E} \int_{\mathbb{R}^n} (f - M_0)^2 p_t = \mathbb{E} \int_{\mathbb{R}^n} (f - M_t)^2 p_t + \mathbb{E}(M_t - M_0)^2.$$

The first summand in (3.6) equals $\mathbb{E} Var_{p_t}(f)$, which is evidently at most $Var_{p_0}(f)$. As for the second summand in (3.6), it equals

$$Var(M_t) = Var(M_f(t, \theta_t)),$$

where the random vector θ_t has the law of $tX + W_t$, where X is distributed according to μ, and where W_t is a Gaussian random vector in $\mathbb{R}^n$ of mean zero and covariance matrix $t \cdot \mathrm{Id}$ that is independent of X. By the subadditivity property of the Poincaré constant (see e.g. Courtade [17]),

(3.7) $$C_P(\theta_t) = C_p(tX + W_t) \leq C_p(tX) + C_p(W_t) = t^2/\lambda_0 + t = \alpha t$$

for $\alpha = 1 + t/\lambda_0$. According to (3.3) and to the Cauchy-Schwartz inequality, for any $\theta \in \mathbb{R}^n$,

$$|\nabla_\theta M_f(t, \theta)| = \sup_{\eta \in S^{n-1}} \int_{\mathbb{R}^n} \langle x - a(t, \theta), \eta \rangle (f(x) - M_f(t, \theta)) p_{t,\theta}(x) dx$$

$$\leq \sqrt{\|A(t, \theta)\|_{op} \cdot Var_{p_{t,\theta}}(f)}.$$

Hence, by (3.7) and the Poincaré inequality for the random vector θ_t,

$$(3.8) \qquad Var(M_f(t,\theta_t)) \leqslant \alpha t \cdot \mathbb{E}|\nabla_\theta M_f(t,\theta_t)|^2 \leqslant \alpha t \cdot \mathbb{E}\left[\|A_t\|_{op} Var_{p_t}(f)\right] \leqslant \alpha \cdot \mathbb{E} Var_{p_t}(f),$$

where we used the second inequality in (3.2) in the last passage, which is equivalent to the bound $\|A_t\|_{op} \leqslant 1/t$. Thus, by (3.6) and (3.8),

$$Var_{p_0}(f) \leqslant \mathbb{E} Var_{p_t}(f) + \alpha \cdot \mathbb{E} Var_{p_t}(f) = (1+\alpha)\mathbb{E} Var_{p_t}(f).$$

$\square$

Corollary 3.2. *For any $A > 0$ and $0 < t < A\lambda_0$,*

$$(3.9) \qquad\qquad\qquad \lambda_0^{-1} \leqslant C(A+1)\mathbb{E}\lambda_t^{-1} \leqslant C(A+1)\frac{\mathbb{E}\sqrt{\|A_t\|_{op}}}{\sqrt{t}},$$

where $C > 0$ is a universal constant.

Proof. E. Milman's theorem [38] states that

$$(3.10) \qquad\qquad\qquad c\lambda_0^{-1} \leqslant \sup_{\varphi \text{ is 1-Lipschitz}} Var_\mu(\varphi)$$

where the supremum runs over all 1-Lipschitz functions $\varphi : \mathbb{R}^n \to \mathbb{R}$, and where $c > 0$ is a universal constant. See [31] for a proof of this theorem that uses *needle decompositions* rather than the regularity theory of the isoperimetric problem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a 1-Lipschitz function with

$$(3.11) \qquad\qquad\qquad Var_\mu(f) \geqslant c\lambda_0^{-1}/2.$$

By Lemma 3.1 and the Poincaré inequality,

$$(3.12) \quad Var_\mu(f) = Var_{p_0}(f) \leqslant (2+A)\mathbb{E} Var_{p_t}(f) \leqslant (2+A)\mathbb{E}\lambda_t^{-1}\int_{\mathbb{R}^n} |\nabla f|^2 p_t \leqslant (2+A)\mathbb{E}\lambda_t^{-1}.$$

The left-hand side inequality in (3.9) follows from (3.11) and (3.12). The right-hand side inequality in (3.9) follows from (3.2). $\square$

Consider the covariance process $A_t = A(t,\theta_t)$ defined for $t \geqslant 0$. Since μ is isotopic we have $A_t = \mathrm{Id}$ for $t = 0$. One of the features of the covariance process is the fact that for $0 < t \leqslant c\psi_n^{-2}/\log n$,

$$(3.13) \qquad\qquad\qquad \mathbb{E}\|A_t\|_{op} \leqslant C.$$

A proof of inequality (3.13) appears in [32]. We remark that Corollary 5.4 in Klartag and Lehec [34] states that (3.13) holds true whenever $t \leqslant c\kappa_n^{-2}/\log n$, where

$$\kappa_n = \sup_X \mathbb{E}\|X_1(X \otimes X)\|_{HS}^2$$

and the supremum runs over all isotropic, log-concave random vectors $X = (X_1,\ldots,X_n) \in \mathbb{R}^n$. The quantity κ_n was used by Eldan [19], and it satisfies

$$(3.14) \qquad\qquad\qquad \kappa_n^2 \leqslant 4\sup_X C_P(X) \leqslant C\psi_n^2$$

for a universal constant $C > 0$, where the supremum runs over all isotropic, log-concave random vectors $X \in \mathbb{R}^n$. Indeed, the right-hand side inequality in (3.14) follows from (1.4)

and (1.5) above. In order to prove the left-hand side inequality in (3.14) we argue as follows: Let X be an isotropic, log-concave random vector, and denote $B = \mathbb{E}X_1 X \otimes X$ and $f(x) = \langle Bx, x \rangle$. Then,

$$\|B\|_{HS}^2 = \mathbb{E}X_1 \langle BX, X \rangle \leqslant \sqrt{\mathbb{E}X_1^2} \cdot \sqrt{Var(f(X))} \leqslant \sqrt{C_P(X) \cdot \mathbb{E}|\nabla f(X)|^2}$$
$$= 2\sqrt{C_P(X) \cdot \mathbb{E}|BX|^2} = 2\sqrt{C_P(X)}\|B\|_{HS}.$$

This implies the left-hand side inequality in (3.14).

Proof of Theorem 1.2. Let μ be an isotropic, log-concave probability measure in $\mathbb{R}^n$ with

$$(3.15) \qquad \qquad \psi_\mu \geqslant \frac{\psi_n}{2}.$$

As we discussed above, for $t = c\psi_n^{-2}/\log n$,

$$(3.16) \qquad \qquad \mathbb{E}\sqrt{\|A_t\|_{op}} \leqslant \sqrt{\mathbb{E}\|A_t\|_{op}} \leqslant C.$$

Since $\lambda_0 = 1/C_P(\mu)$, from (1.4) and (1.5) we know that

$$(3.17) \qquad \qquad t \leqslant c'\psi_n^{-2} \leqslant \tilde{c}\psi_\mu^{-2} \leqslant \bar{c}\lambda_0.$$

We now apply Corollary 3.2, together with (3.16) and (3.17), and obtain

$$(3.18) \qquad \qquad \lambda_0^{-1} \leqslant \frac{C}{\sqrt{t}} \leqslant \tilde{C}\psi_n\sqrt{\log n}.$$

Consequently, by using (1.4), (3.15) and (3.18),

$$\psi_n^2 \leqslant 4\psi_\mu^2 \leqslant \tilde{C}\lambda_0^{-1} \leqslant \bar{C}\psi_n\sqrt{\log n}.$$

This implies that $\psi_n \leqslant C\sqrt{\log n}$. $\qquad \qquad \square$

4. REMARKS

One point in the proof above which seems counter-intuitive is the fact that if the eigen-function f satisfies

$$(4.1) \qquad \qquad \left|\int_{\mathbb{R}^n} f(x)x\,d\mu(x)\right|^2 = o\left(\|Cov(\mu)\|_{op}\right),$$

then our estimates become *better*. Here μ is an isotropic, regular, log-concave probability measure in $\mathbb{R}^n$, and $f \in L^2(\mu)$ is an eigenfunction satisfying $Lf = -\lambda(\mu)f$ and $\|f\|_{L^2(\mu)} = 1$. Intuitively, strong correlation between the eigenfunction and a linear function should improve estimates towards the KLS conjecture, rather than the other way around. One could speculate that analysis of the H^{-1}-norm of linear functionals, which played a crucial role in Klartag and Lehec [34], could help us understand this point. The $H^{-1}(\mu)$-norm may be defined, for $f \in L^2(\mu)$ with $\int_{\mathbb{R}^n} f\,d\mu = 0$, via

$$\|f\|_{H^{-1}(\mu)} := \sup\left\{\int_{\mathbb{R}^n} fg\,d\mu; g \in L^2(\mu) \text{ is locally Lipschitz with } \int_{\mathbb{R}^n} |\nabla g|^2 d\mu \leqslant 1\right\}.$$

See [4, 30] for basic properties of the $H^{-1}(\mu)$-norm and for the inequality

$$\lambda(\mu) \cdot \|f\|^2_{H^{-1}(\mu)} \leq \|f\|^2_{L^2(\mu)}.$$

Here is a certain version of the improved log-concave Lichnerwicz inequality:

Proposition 4.1. *Let $t > 0$ and assume that μ is a probability measure in $\mathbb{R}^n$ that is regular and t-uniformly log-concave. Assume furthermore that μ is centered, i.e., $b_\mu = \int_{\mathbb{R}^n} x \, d\mu(x) = 0$. Denote*

$$R = \sup_{\theta \in S^{n-1}} \|\langle \cdot, \theta \rangle\|^2_{H^{-1}(\mu)}.$$

Then $\lambda(\mu) \geq (t/R)^{1/3}$.

Proof. We use the notation from the proof of Theorem 1.3. From (2.7) and (2.8) we obtain

$$\int_{\mathbb{R}^n} \left[(\nabla^2 \psi) \nabla f \cdot \nabla f \right] d\mu \leq \lambda^3 \left| \int_{\mathbb{R}^n} f(x) x \, d\mu(x) \right|^2 = \lambda^4 \sup_{\theta \in S^{n-1}} \left| \int_{\mathbb{R}^n} \frac{f(x)}{\sqrt{\lambda}} \langle x, \theta \rangle d\mu(x) \right|^2$$

$$\leq \lambda^4 \sup_{\theta \in S^{n-1}} \|\langle \cdot, \theta \rangle\|^2_{H^{-1}(\mu)} = \lambda^4 R.$$

Hence, from (2.12) we obtain $t\lambda \leq \lambda^4 R$, which yields $\lambda \geq (t/R)^{1/3}$. $\qquad \square$

Remark 4.2. *As explained in* [4, 34]*, for any $f \in L^2(\mu)$ with $\int_{\mathbb{R}^n} f \, d\mu = 0$ we have*

$$\|f\|^2_{H^{-1}(\mu)} = -\int_{\mathbb{R}^n} (L^{-1} f) f \, d\mu,$$

where here L is the unique self-adjoint extension of the operator defined on $\mathscr{F}_\mu$ via (2.3). Observe that $Lx = -\nabla \psi$, and that the vector-valued functions $\nabla \psi$ and x are μ-tempered. Therefore,

$$(4.2) \qquad -\int_{\mathbb{R}^n} \langle L^{-1}(\nabla \psi), \nabla \psi \rangle d\mu = \int_{\mathbb{R}^n} \langle x, \nabla \psi \rangle e^{-\psi} = -\sum_{i=1}^n \int_{\mathbb{R}^n} x_i \partial_i (e^{-\psi}) = n \cdot \int_{\mathbb{R}^n} e^{-\psi} = n.$$

The H^{-1}-inequality [4, 30] *states that for any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ with $\int \nabla f \, d\mu = 0$,*

$$(4.3) \qquad Var_\mu(f) \leq \sum_{i=1}^n \left\| \partial^i f \right\|^2_{H^{-1}(\mu)} =: \left\| \nabla f \right\|^2_{H^{-1}(\mu)},$$

where $Var_\mu(f) = \int f^2 d\mu - (\int f \, d\mu)^2$. Hence, from (4.2) and (4.3) we obtain an alternative proof of the varentropy inequality of Bobkov and Madiman [6] *and Nguyen* [39]*, which is the inequality:*

$$Var_\mu(\psi) \leq \|\nabla \psi\|^2_{H^{-1}(\mu)} = n.$$

There is more than one way to deduce Theorem 1.2 from the improved log-concave Lichnerowicz inequality. The following variant of the integrated Bochner formula shows another connection between the Laplace operator associated with p_0 and the one associated with p_t, in addition to the connection established in Corollary 3.2 above by using 1-Lipschitz functions.

Proposition 4.3 ("Localized Bochner formula"). *Let μ be a probability measure on $\mathbb{R}^n$ with a regular, log-concave probability density $\rho = e^{-\psi}$. Let $u \in \mathscr{F}_\mu$, $t > 0$, and consider the random probability density p_t defined in Section 3. Write μ_t for the regular, log-concave probability measure on $\mathbb{R}^n$ whose density is p_t and abbreviate $L_t = L_{\mu_t}$ and $L = L_\mu$. Then with probability one, $u \in \mathscr{F}_{\mu_t}$ and*

$$\int_{\mathbb{R}^n} (Lu)^2 \, d\mu + t \int_{\mathbb{R}^n} |\nabla u|^2 \, d\mu = \mathbb{E} \int_{\mathbb{R}^n} (L_t u)^2 \, d\mu_t.$$

Proof. We know that $d\mu_t/d\mu$ is a Gaussian density, hence the fact that $u \in \mathscr{F}_\mu$ implies that $u \in \mathscr{F}_{\mu_t}$. Write $p_t = e^{-\psi_t}$ and apply the Bochner formula for the measure μ_t to obtain

$$(4.4) \qquad \int_{\mathbb{R}^n} (L_t u)^2 \, d\mu_t = \int_{\mathbb{R}^n} \|\nabla^2 u\|^2_{HS} \, d\mu_t + \int_{\mathbb{R}^n} \langle (\nabla^2 \psi_t) \nabla u, \nabla u \rangle \, d\mu_t.$$

From formula (3.1) we know that

$$\nabla^2 \psi_t(x) = \nabla^2 \psi(x) + t \cdot \mathrm{Id} \qquad\qquad (x \in \mathbb{R}^n).$$

Therefore, by combining (4.4) with (3.5) and Fubini's theorem,

$$\mathbb{E} \int_{\mathbb{R}^n} (L_t u)^2 \, d\mu_t = \mathbb{E} \int_{\mathbb{R}^n} \|\nabla^2 u\|^2_{HS} \, d\mu_t + \mathbb{E} \int_{\mathbb{R}^n} \langle (\nabla^2 \psi) \nabla u, \nabla u \rangle \, d\mu_t + t \cdot \mathbb{E} \int_{\mathbb{R}^n} |\nabla u|^2 \, d\mu_t$$

$$= \int_{\mathbb{R}^n} \|\nabla^2 u\|^2_{HS} \, d\mu + \int_{\mathbb{R}^n} \langle (\nabla^2 \psi) \nabla u, \nabla u \rangle \, d\mu + t \cdot \int_{\mathbb{R}^n} |\nabla u|^2 \, d\mu.$$

The proposition now follows from the Bochner formula (2.5) applied for the measure μ.

$\square$

5. Appendix

Write γ_s for the density of a Gaussian random vector of mean zero and covariance $s \cdot \mathrm{Id}$ in $\mathbb{R}^n$, i.e., $\gamma_s(x) = (2\pi s)^{-n/2} \exp\left(-|x|^2/(2s)\right)$ for $x \in \mathbb{R}^n$. The following lemma allows us to interchange Gaussian convolution and multiplication by a Gaussian density.

Lemma 5.1. *For $f : \mathbb{R}^n \to \mathbb{R}$ and $r > 0$ denote $S_r f(x) = r^n f(rx)$, the scaling of f when viewed as a density. Let $s, t > 0$ and set*

$$p = \frac{st}{s+t}, \qquad q = \frac{t^2}{s+t} \qquad and \qquad r = \frac{t}{s+t}.$$

Then for any measurable function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f\gamma_t$ is integrable,

$$(f\gamma_t) * \gamma_s = S_r \left[(f * \gamma_p)\gamma_q \right].$$

Proof. Let $x \in \mathbb{R}^n$ and set $X = px/s$. Then,

$$[(f\gamma_t) * \gamma_s](x) = (2\pi)^{-n}(st)^{-n/2} \int_{\mathbb{R}^n} f(y) e^{-\frac{|y|^2}{2t} - \frac{|x-y|^2}{2s}} dy$$

$$= (2\pi)^{-n}(st)^{-n/2} e^{\frac{|x|^2(-1/s+p/s^2)}{2}} \int_{\mathbb{R}^n} f(y) e^{-\frac{|y-px/s|^2}{2p}} dy$$

$$= (2\pi)^{-n}(st)^{-n/2} e^{-\frac{|X|^2}{2q}} \int_{\mathbb{R}^n} f(y) e^{-\frac{|y-X|^2}{2p}} dy = r^n \left[(f * \gamma_p)\gamma_q\right](X).$$

$\square$

Corollary 5.2. *Let $s, t > 0$, and let μ be an absolutely-continuous probability measure on $\mathbb{R}^n$ which is $1/t$-uniformly log-concave. Then the convolution $\mu * \gamma_s$ is $1/(t+s)$-uniformly log-concave.*

Proof. We say that f is more log-concave than γ_t if f/γ_t is log-concave, i.e., if f is $1/t$-uniformly log-concave. Write $\rho\gamma_t$ for the density of μ, thus ρ is log-concave. Define p, q, r as in Lemma 5.1. By the Prékopa-Leindler inequality, the function $\rho * \gamma_p$ is log-concave, where the convolution is well-defined by Lemma 5.1. The density of $\mu * \gamma_s$ is $(\rho\gamma_t) * \gamma_s$, which by Lemma 5.1 is more log-concave than

$$S_r \gamma_q = \gamma_{s+t}.$$

Indeed, we compute that

$$S_r \gamma_q(x) = (2\pi q/r^2)^{-n/2} \exp\left(-|rx|^2/(2q)\right) = \gamma_{q/r^2} = \gamma_{s+t}.$$

$\square$

Proof of Lemma 2.1. Let $0 < \delta < 1$ be a sufficiently small number, and write ρ for the log-concave density of μ. The measure $\nu = \nu_\delta$ is obtained by convolving ρ with γ_δ, then multiplying the resulting density by $\gamma_{1/\delta}$, and then normalizing to obtain a probability density. The Prekopá-Leindler inequality shows that ν is a log-concave measure, and ν clearly has a smooth, positive density $e^{-\psi_\delta}$. We claim that

(5.1) $$\delta \leqslant \nabla^2 \psi_\delta(x) \leqslant \delta + 1/\delta \qquad \text{for all } x \in \mathbb{R}^n.$$

The left-hand side inequality in (5.1) follows from the fact that ν is obtained by multiplying a log-concave density by $\gamma_{1/\delta}$ and then normalizing. For the right-hand side inequality, we write

(5.2) $$e^{-\psi_\delta(x)} = e^{-(\delta+1/\delta)|x|^2/2} \int_{\mathbb{R}^n} e^{\frac{\langle x, y \rangle}{\delta} - \frac{|y|^2}{2\delta}} \rho(y) dy / Z_\delta$$

where $Z_\delta > 0$ is a normalizing constant. The logarithm of the integral in (5.2) is a convex function of x, leading to the right-hand side inequality in (5.1). The fact that the function ψ_δ, as well as each of its partial derivatives, grows at most polynomially at infinity is proven in [33, Lemma 2.2]. Thus ν_δ is a regular, log-concave probability measure for any $0 < \delta < 1$. Next, consider a smooth function f with

$$\int_{\mathbb{R}^n} f^2 d\mu - \left(\int_{\mathbb{R}^n} f d\mu\right)^2 \geqslant (C_P(\mu) - \varepsilon/2) \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.$$

By the dominated convergence theorem we may replace the integrals with respect to μ by integrals with respect to ν_δ while accumulating only a tiny error. This shows that for a sufficiently small $\delta > 0$,

$$C_P(\nu_\delta) \geqslant C_P(\mu) - \varepsilon.$$

The dominated convergence theorem also shows that the covariance matrix of $\nu = \nu_\delta$ can be made arbitrarily close to that of ν, for a sufficiently small $\delta > 0$. Finally, if μ is t-uniformly log-concave, then Corollary 5.2 implies that ν_δ is $[\delta + t/(1 + t\delta)]$-uniformly log-concave. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Lemma 2.2. Let $\varphi_k \in L^2(\mu)$ be an eigenfunction of L_μ corresponding to some eigenvalue $-\lambda_k$. Since L is an elliptic operator, the function φ_k is smooth. As in Remark 2.7 in Klartag and Putterman [33], for any $k \geqslant 0$ the smooth function $\tilde{\varphi}_k = \varphi_k \cdot \sqrt{\rho} \in L^2(\mathbb{R}^n)$ is an eigenfunction of the Schrödinger operator $-\Delta + V$ for

$$V = |\nabla \psi|^2/4 - \Delta \psi/2,$$

where we recall that $\rho = e^{-\psi}$. The potential $V(x)$ is a smooth function that tends to infinity as $x \to \infty$. In fact, since μ is regular then

$$V(x) \geqslant \varepsilon|x| - 1/\varepsilon$$

for some $\varepsilon > 0$ depending on μ. It is known that any eigenfunction $\tilde{\varphi}_k$ of the Schrödinger operator $-\Delta + V$ decays exponentially at infinity, see Agmon [1, Chapter 5], Carmona [12], Combes and Thomas [16], O'Connor [40] or Reed and Simon [41, Theorem XIII.70]. Moreover, since μ is regular, the potential V and each of its partial derivatives grows at most polynomially at infinity. Hence both functions $\tilde{\varphi}_k$ and $\Delta \tilde{\varphi}_k = (V - \lambda_k)\tilde{\varphi}_k$ decay exponentially at infinity. This implies that each partial derivative of $\tilde{\varphi}_k$ decays exponentially at infinity; to see this, consider a unit ball centered at a faraway point $x \in \mathbb{R}^n$, and express $\tilde{\varphi}_k$ via Green's representation for solutions to the Poisson equation in this unit ball (e.g. [21, Section 2.2.4]). For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers, in the notation of [33, Section 2],

$$(5.3) \qquad\qquad \partial^\alpha \varphi_k = \partial^\alpha(\tilde{\varphi}_k e^{\psi/2}) = \sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta \tilde{\varphi}_k \cdot \partial^{\alpha - \beta} e^{\psi/2},$$

where we used the Leibnitz rule. Each of the summands in (5.3) is some coefficient multiplied by the product of three factors: the exponentially decaying function $\partial^\beta \tilde{\varphi}_k$, a certain expression involving derivatives of ψ that grows at most polynomially, and the function $e^{\psi/2}$. This implies that $\varphi_k \in \mathscr{F}_\mu$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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