Exercise 3: June 27

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Exercise 1 (Randomized Symmetry Breaking). For an n-vertex graph $G = (V, E)$, and a vertex $u \in V$, let $N(u)$ be the neighbors of u in G, and let $deg(u) = |N(u)|$ be its degree. For integers $1 \leq \alpha \leq \beta$, a subset $V' \subseteq V$ is an (α, β) ruling-set if (i) $d_G(u, v) \geq \alpha$ for every $u, v \in V'$, and (ii) for every $u \in V$, there exists $v \in V'$ such that $d_G(u, v) \leq \beta$.

Consider the following 1-round randomized LOCAL algorithm: each vertex v picks a number r_v u.a.r in $[0, 1]$, and joins a set R if it is a local minima, i.e., if $r_v < r_u$ for any neighbor $u \in N(v)$. We next prove two properties of this algorithm.

(1a) Show that for each v it holds that $R \cap N^+(v) \neq \emptyset$ with probability at least $(deg(v)+1)/(deg(v)+deg_{max})$, where $N^+(v) = N(v) \cup \{v\}$ and deg_{max} is the maximum degree among nodes in $N^+(v)$.
(1b) Assume that G is a nearly regular graph with all degrees in $[\Delta, 2\Delta]$. Show that w.h.p. R is a $(2, O(\log n))$ ruling set. **Bonus**: prove the above claim for any n-vertex graph G, i.e., without assuming that G is nearly regular.

Exercise 2 (Strong Network Decomposition). Let $\mathcal{A}$ be the deterministic LOCAL algorithm for (c, d) *weak* network decomposition of [RG20], and let $\mathcal{B}$ be the deterministic sequential algorithm for (c, d) *strong* network decomposition where $c, d = O(\log n)$. Both of these algorithms were presented in class. **Show**: There is a deterministic LOCAL algorithm for computing for computing $(O(\log n), O(\log n))$ strong network decomposition using $poly(\log n)$ rounds. You are allowed to use Alg. $\mathcal{A}$ and $\mathcal{B}$ in a black-box manner.

Exercise 3 (Low Diameter Ordering). The last exercise illustrates another combinatorial application of network decomposition.

Definition 3.1 *Given a n-node graph $G = (V, E)$, a $d(n)$-diameter ordering of G is an assignment of unique labels*[1] *to all nodes V such that any path P on which the labels are increasing along P, any two nodes of P are within a distance $d(n)$ in the graph G.*

Show: any n-vertex graph has a $d(n)$-diameter ordering with $d(n) = O(\log^2 n)$.

[1] The label is simply a unique identifier to the vertices of $O(\log n)$ bits, i.e., a number in $[1, poly(n)]$.