Succinct Graph Structures and Their Applications **Spring 2024**

Home Exam : July 30

Lecturer: Merav Parter

Instructions: The exam consists of 4 problem sets, each worth 25 points. In case needed (and unless stated otherwise), you are allowed to use constructions that we showed in class as a black-box, but still required to add full proofs. Typing in LaTeX (and such) is highly recommended, but not a must (in case you do not have time). Please submit your solutions through email to merav.parter@weizmann.ac.il by midnight of July 31, 12am. Good luck!

Labeling Schemes

Exercise 1. Let T be an n-vertex undirected tree, where each vertex in T has a unique identifier $id(v)$ with $O(\log n)$ bits. (a) show an $O(\log^2 n)$-size labeling scheme for computing the LCA (lowest common ancestor) between two vertices u and v. I.e., design labels $L(u)$ of $O(\log^2 n)$ bits such that given $L(u)$ and $L(v)$, one can compute the identifier of the LCA of u and v in the tree T.
(b) For every three vertices a, b, c, let $Center(a, b, c)$ denote the center of these vertices, namely, a vertex z such that the three tree paths from z to a, b, c are edge-disjoint. Show that one can assign a $O(\log^2 n)$-bit labels such that given the labels of any three vertices a, b, c, one can deduce the identifier of their center $Center(a, b, c)$.

Spanners

Exercise 2. We showed in class the Baswana-Sen algorithm for computing 3-spanners for unweighted undirected graph. Extend this algorithm to provide a randomized almost linear time algorithm for computing 3-spanners for n-vertex *weighted* graph. You are required to describe the entire algorithm and its analysis.

Fault Tolerant Graph Structures

Exercise 3. For an undirected graph G, a subgraph $G' \subseteq G$ and vertex pair $u, v \in V$, define $\texttt{conn}(u, v, G') = 1$ iff u and v are connected in G'. A subgraph $H \subseteq G$ is an *f-edge FT-connected* subgraph if for every $u, v \in V$ and $F \subseteq E$, $|F| \leq f$, it holds that:

$$\texttt{conn}(u, v, H \setminus F) = \texttt{conn}(u, v, G \setminus F) \ .$$

Show an $O(f \cdot m)$-time algorithm that given any n-vertex graph G and parameter f, constructs an *f-edge FT-connected* subgraph $H \subseteq G$ with at most $(f + 1) \cdot (n - 1)$ edges.

Exercise 4. Consider an unweighted n-vertex graph $G = (V, E)$ with diameter D. (a) Show that for every $s, t \in V$ and $F \subseteq E$, if s and t are connected in $G \setminus F$, then $\texttt{dist}(s, t, G \setminus F) = O(|F|D)$. (b) For a given source s, a 2-FT BFS tree $H \subseteq G$ is a G-subgraph satisfying that $\texttt{dist}(s, t, H \setminus F) = \texttt{dist}(s, t, G \setminus F)$ for every $t \in V$ and every pair of edges $F \subseteq E$, $|F| \leq 2$. Use (a) to show that one can compute a 2-FT BFS tree $H \subseteq G$ with $O(D^2 n)$ edges.