Lecture 10: Pesin Charts and Applications

Setup: $T: M \to M$ is a C^2 diffeomorphism on a Riemannian manifold and $\Lambda \subseteq M$ is a hyperbolic set:

(1) Λ is compact and T-invariant

(2) Oseledets Decomposition: $\forall x \in \Lambda$, $T_x M = E^s(x) \oplus E^u(x)$, $E^{s/u}(x) =$ linear spaces

- Invariance: $(dT)_x E^s(x) = E^s(Tx)$, $(dT)_x E^u(x) = E^u(Tx)$
- Uniform Expansion / Contraction: $\exists\, C > 0$, $0 < \lambda < 1$ s.t. $\forall x \in \Lambda$
  - $\forall \vec{v}^s \in E^s(x)$, $\forall n \geq 0$ $\|(dT^n)_x \vec{v}^s\| \leq C\lambda^n \|\vec{v}^s\|$
  - $\forall \vec{v}^u \in E^u(x)$, $\forall n \geq 0$ $\|(dT^{-n})_x \vec{v}^u\| \leq C\lambda^n \|\vec{v}^u\|$

  ($\Rightarrow$ $\forall \vec{v}^u \in E^u(x)$, $\forall n \geq 0$ $\|(dT^n)_x \vec{v}^u\| \geq C^{-1}\lambda^{-n} \|\vec{v}^u\|$.)

In this lecture we analyze the dynamics of $T: \Lambda \to \Lambda$ and show:

(a) T has exponential sensitivity to initial conditions at each $x \in \Lambda$

(b) Stable manifold theorem

The key to these results is a convenient system of local coordinates charts called Pesin charts which transform $T^n = T \circ \cdots \circ T$ near x into $F_{T^{n-1}(x)} \circ F_{T^{n-2}(x)} \circ \cdots \circ F_{Tx} \circ F_x$ where $F_{T^k(x)}: \mathbb{R}^d \to \mathbb{R}^d$ are perturbations of linear hyperbolic maps.

Everything works in arbitrary dimension, but we'll stick to the case $\dim M = 2$, which contains all the essential ideas, but is easier to explain.

Pesin Charts

"Diagonalizing" dT on Λ: We construct linear change of cordinates in T_xM, which bring $dT_x : T_xM \to T_{T(x)}M$ to diagonal form.

We stick to $\dim M = 2$ (the general case is similar, but more tedious). In the two dimensional case, $\exists\, 0<\lambda<1$ s.t. $\forall x\in\Lambda$

- $E^s(x) = \operatorname{Span}\{\vec{e}_s(x)\}$, $\|\vec{e}_s(x)\| = 1$
- $E^u(x) = \operatorname{Span}\{\vec{e}_u(x)\}$, $\|\vec{e}_u(x)\| = 1$
- $\|(dT^n)_x \vec{e}_s(x)\| \le C\lambda^n$ $(n\ge 1)$
- $\|(dT^{-n})_x \vec{e}_u(x)\| \le C\lambda^n$ $(n\ge 1)$

Fix some $\chi>0$ s.t. $e^{\chi}\lambda<1$ and define (for fixed $x\in\Lambda$):

$$s_\chi(x) := \sqrt{2}\left(\sum_{n=0}^{\infty} e^{2n\chi}\|(dT^n)_x e^s(x)\|^2\right)^{1/2}$$

$$u_\chi(x) := \sqrt{2}\left(\sum_{n=0}^{\infty} e^{2n\chi}\|(dT^{-n})_x e^s(x)\|^2\right)^{1/2}$$

(The sum converge because the summand is $O(e^{2n\chi}\lambda^{2n})$, $e^{\chi}\lambda<1$.)

Fix $x\in\Lambda$ and define a linear map $C(x):\mathbb{R}^2 \to T_xM$ by

$$\boxed{C(x)\begin{pmatrix}1\\0\end{pmatrix} := s_\chi(x)^{-1}\vec{e}^{\,s}(x),\quad C(x)\begin{pmatrix}0\\1\end{pmatrix} := u_\chi(x)^{-1}\vec{e}^{\,u}(x)}$$

Thm ("Oseledets–Pesin Reduction"): $\exists\, C_0>0$ s.t. $\forall x\in\Lambda$,

(1) $C(Tx)^{-1}\circ(dT)_x\circ C(x) = \begin{pmatrix}\lambda_\chi(x) & 0\\ 0 & \mu_\chi(x)\end{pmatrix}$ ← diagonal!

(2) $C_0^{-1} < |\lambda_\chi(x)| < e^{-\chi} < 1$ $\quad 1 < e^{\chi} < |\mu_\chi(x)| < C_0$ ← uniformly hyperbolic!

<u>Proof</u>: The key observation is that since

$$(dT)_x\left[\mathrm{Span}\{\vec{e}^s(x)\}\right] = dT_x(E^s(x)) = E^s(Tx) = \mathrm{Span}\{\vec{e}^s(Tx)\},$$

it must be the case that

$$\boxed{(dT)_x\, \vec{e}^s(x) = \pm \|(dT)_x \vec{e}^s(x)\|\, e^s(Tx).}$$

Therefore: $C_\chi(Tx)^{-1}(dT)_x C_\chi(x)\begin{pmatrix}1\\0\end{pmatrix} = \pm C_\chi(Tx)^{-1}\left[s_\chi(x)^{-1}\|(dT)_x\vec{e}^s(x)\| e^s(Tx)\right]$

$$= \pm \frac{s_\chi(Tx)}{s_\chi(x)} \|(dT)_x\, \vec{e}^s(x)\| \begin{pmatrix}1\\0\end{pmatrix}.$$

So $\begin{pmatrix}1\\0\end{pmatrix}$ is an eigenvector with e.v. $\lambda_\chi(x) := \pm \frac{s_\chi(Tx)}{s_\chi(x)} \|(dT)_x\, \vec{e}^s(x)\|$.

Similarly, $\begin{pmatrix}0\\1\end{pmatrix}$ is an eigenvector, with eigenvalue

$$\mu_\chi(x) := \pm \frac{u_\chi(Tx)}{u_\chi(x)} \|(dT)_x\, \vec{e}^u(x)\|$$

This shows that $\boxed{C_\chi(Tx)^{-1} dT_x C_\chi(x) = \begin{pmatrix}\lambda_\chi & 0\\ 0 & \mu_\chi\end{pmatrix}}$

Next we <u>estimate the eigenvalues</u>:

$$s_\chi(x)^2 = 2\sum_{n=0}^{\infty} e^{2n\chi}\|dT_x^n\, \vec{e}^s(x)\|^2 > 2\sum_{n=1}^{\infty} e^{2n\chi}\|dT_x^n \vec{e}^s(x)\|^2$$

$$= 2\sum_{n=0}^{\infty} e^{2(n+1)\chi}\|(dT^n)_{Tx}\,(dT_x\vec{e}^s(x))\|^2$$

$$= \|dT_x\,\vec{e}^s(x)\|^2\cdot e^{2\chi}\cdot \underbrace{2\sum_{n=0}^{\infty} e^{2n\chi}\|dT_x^n e^s(Tx)\|^2}_{= s_\chi(Tx)^2}$$

$$= e^{2\chi}\|dT_x\,\vec{e}^s(x)\|^2 s_\chi(Tx)^2.$$

Thus $s_\chi(Tx)^2 \| dT_x \vec{e}^s(x)\|^2 < e^{-2\chi} s_\chi(x)^2$, whence

$$\boxed{|\lambda_\chi(x)| = \frac{s_\chi(Tx)\, \| dT_x \vec{e}^s(x)\|}{s_\chi(x)} < e^{-\chi}}$$

Next, we bound $|\lambda_\chi(x)|$ from below. Observe first that by the compactness of M and the C^1-smoothness of T,

$$M := \sup\{ \|dT_x\|, \|(dT^{-1})_x\| : x \in M\} < \infty.$$

$$\begin{aligned}
s_\chi(x)^2 &\equiv 2 \sum_{n=0}^{\infty} e^{2n\chi} \| dT_x^n \vec{e}^s(x)\|^2 \\
&\leq 2 \left(1 + \sum_{n=1}^{\infty} e^{2n\chi} \| dT_{Tx}^{n-1} \vec{e}^s(Tx)\|^2 \cdot \|dT_x e^s(x)\|^2\right) \\
&\leq 2\left(1 + e^{2\chi} \|dT_x e^s(x)\|^2 \sum_{n=1}^{\infty} e^{2(n-1)\chi} \|dT_x^{n-1} e^s(Tx)\|^2\right) \\
&\leq (1 + e^{2\chi} M^2)\, 2 \sum_{n=0}^{\infty} e^{2n\chi} \|dT_{Tx}^n \vec{e}^s(Tx)\|^2 \\
&\leq (1 + e^{2\chi} M^2)\, s_\chi(Tx)^2.
\end{aligned}$$

Thus $$\lambda_\chi(x)^2 = \frac{s_\chi(Tx)^2}{s_\chi(x)^2} \|dT_x e^s(x)\|^2 \geq (1+e^{2\chi}M^2)^{-1} \cdot \frac{1}{\sup_x \|dT_x^{-1}\|^2}$$

$$\geq \left[M^2(1+e^{2\chi}M^2)\right]^{-1} = \text{global const.}$$

Similarly one shows that

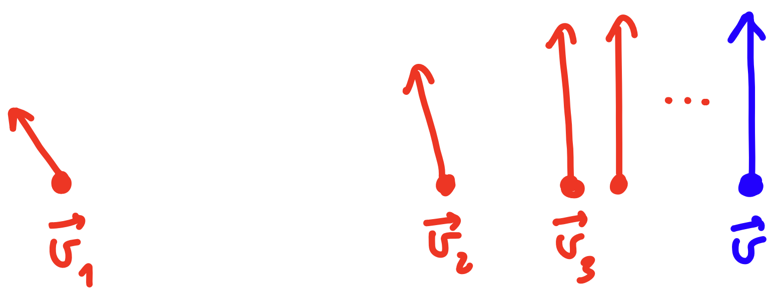
$$e^{\chi} \leq u_\chi(x) \leq \text{const} \qquad \square$$

For the next lemma, it is useful to assume that our manifold is isometrically embedded in $\mathbb{R}^N$ for some $N \gg d$. (It's a general theorem that such an embedding exists.)

In this case, the tangent vectors to M are vectors in $\mathbb{R}^N$

We say that $\vec{v}_n \longrightarrow \vec{v}$ if

- (base point of $\vec{v}_n$) $\longrightarrow$ (base point of $\vec{v}$)
- (direction of $\vec{v}_n$) $\longrightarrow$ (direction of $\vec{v}$)
- (size of $\vec{v}_n$) $\longrightarrow$ (size of $\vec{v}$)



<u>Continuity Lemma</u>: Suppose Λ is a hyperbolic set for a C^1 diffeomorphism on a compact Riemannian manifold. Suppose $x_n \in \Lambda$, and $x_n \xrightarrow[n\to\infty]{} x$. Then:

(1) $x \mapsto E^u(x), E^s(x)$ are continuous on Λ. Specifically:

$$\exists \sigma_n^u, \sigma_n^s \in \{\pm 1\} \text{ s.t. } \sigma_n^u \vec{e}^u(x_n) \xrightarrow[n\to\infty]{} \vec{e}^u(x), \quad \sigma_n^s e^s(x_n) \longrightarrow e^s(x)$$

(2) Let $I_n = \begin{pmatrix} \sigma_n^u & 0 \\ 0 & \sigma_n^s \end{pmatrix}$, then $C_\chi(x_n) I_n \xrightarrow[n\to\infty]{} C_\chi(x)$

(3) $\|C_\chi(x)\|, \|C_\chi(x)^{-1}\|$ are continuous and uniformly bounded away from zero and infinity on Λ.

(4) In fact, $\|C_\chi(x)\| \leq 1$ on Λ.

Proof. Suppose $x_n \in \Lambda$ and $x_n \to x$. Since Λ is compact, $x \in \Lambda$.

Recall that $\vec{e}^s(x_n)$ are unit vectors. Therefore $\{\vec{e}^s(x_n)\}_{n=1}^{\infty}$ is precompact. Fix some $n_k \to \infty$ s.t. $\vec{e}^s(x_{n_k}) \xrightarrow[k\to\infty]{} \vec{v}$.

- Since $x_n \to x$, $\vec{v} \in T_x M$
- Since $\|dT^N_{x_{n_k}} e^s(x_{n_k})\|_{T^N(x_{n_k})} \leq C\lambda^N$ and T is continuously different.,

  $\forall N \geq 0 \left(\|dT^N_x \vec{v}\|_{T^N(x)} \leq C\lambda^N \right)$. So $\vec{v}$ cannot have a component in $E^u(x)$ (otherwise $\|dT^N_x \vec{v}\|$ would explode, not shrink). So $\vec{v} \in E^s(x)$. Also

$$\|\vec{v}\| = \lim_{k\to\infty} \|\vec{e}^s(x_{n_k})\| = 1,$$

therefore $\vec{v} \in \pm \vec{e}^s(x)$.

Thus: any limit point of $\vec{e}^s(x_n)$ equals $\pm\vec{e}^s(x)$.

Set $\sigma^s_n = \mathrm{sgn}\left[\cos \measuredangle(\underbrace{e^s(x_n), e^s(x)}_{\text{viewed as direction in } \mathbb{R}^N})\right]$. Then $\sigma^s_n \vec{e}^s(x_n) \to \vec{e}^s(x)$

(because on subsequences s.t. $\vec{e}^s(x_n) \to \pm\vec{e}^s(x)$, $\sigma^s_n \to \pm 1$).

(2) $x \mapsto s_x(x) := \sqrt{2}\left(\sum_{n=0}^{\infty} e^{2n\chi} \|(dT^n)_x \vec{e}^s(x)\|^2\right)$ is continuous on Λ because, by (1), the summands are continuous in x, and the series converges uniformly on Λ (it's dominated by $\sum C\lambda^n e^{\chi n}$).

Similarly $x \mapsto u_x(x)$ is continuous on Λ.

Since $s_x, u_x \geq \sqrt{2}$, $x \mapsto u_x(x)^{-1}, s_x(x)^{-1}$ are continuous.

Thus $C_x(x_n) I_n \binom{1}{0} = s_x(x_n)^{-1} \cdot \sigma_n^s \vec{e}^s(x_n)$

$$\longrightarrow s_x(x)^{-1} \vec{e}^s(x) = C_x(x) \binom{1}{0}$$

Similarly, $C_x(x_n) I_n \binom{0}{1} \longrightarrow C_x(x) \binom{0}{1}$.

(3) This follows from (2).

(4) Define a new inner product $\langle \cdot, \cdot \rangle_x^*$ on $T_x M$ by:

- $\| e^s(x) \|_x^* := s_x(x)$
- $\| e^u(x) \|_x^* := u_x(x)$
- $\langle e^u(x), e^s(x) \rangle_x^* = 0$.

(This inner product is called the Lyapunov inner product.)

<u>Claim</u>: $\| \cdot \|_x^* \geq \| \cdot \|_x$

<u>Proof</u>. $\forall a, b \in \mathbb{R}$

$$\| a e^s + b e^u \|_x^* = \sqrt{a^2 s_x(x)^2 + b^2 u_x(x)^2} \geq \sqrt{2(a^2+b^2)} \quad (\because s_x, u_x \geq \sqrt{2})$$

$$= \sqrt{a^2 + \underbrace{(a^2+b^2)}_{\geq 2|a||b|} + b^2} \geq \sqrt{(|a|+|b|)^2} = |a| + |b|$$

$$= \| a e^s \|_x + \| b e^u \|_x \geq \| a e^s + b e^u \|_x .$$

<u>Claim</u>: $\| C_x(x) \| \leq 1$:

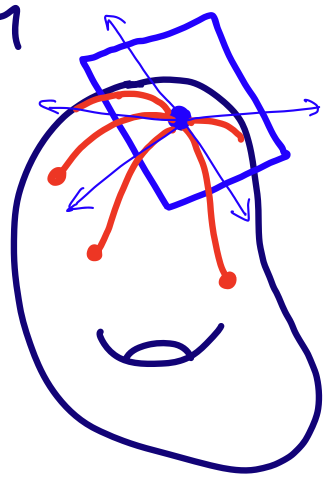
$$\| C_x(x) \binom{a}{b} \|_x \leq \| C_x(x) \binom{a}{b} \|_x^* = \| s_x(x)^{-1} a e^s(x) + u_x(x)^{-1} b e^u(x) \|_x^*$$

$$= \sqrt{\left(\| (s_x(x)^{-1} a) e^s(x) \|_x^* \right)^2 + \left(\| (u_x(x)^{-1} b) e^u(x) \|_x^* \right)^2}$$

$$= \sqrt{a^2 + b^2} = \| \binom{a}{b} \|_{\mathbb{R}^2} .$$

"Diagonalizing" $T:\Lambda\to\Lambda$ We construct local coordinate charts which bring $T:\Lambda\to\Lambda$ near x to the form of a "perturbed" linear hyperbolic map (which depends on x).

The exponential map $\exp_x : T_xM \to M$ is defined as follows:



$\exp_x(\vec{v}) = \gamma_{\vec{v}}(\|\vec{v}\| t)$, where

$\gamma_{\vec{v}}(t)$ is the geodesic from x in direction $\frac{\vec{v}}{\|\vec{v}\|}$.

Facts from Differential Geometry: If M is a compact Riemannian manifold, then

- $\exists\, r(M)>0$, called the injectivity radius, s.t.

$$\exp_x : \{\vec{v}\in T_xM : \|\vec{v}\|<r(M)\} \to M$$

is injective, differentiable, and

$$\exp_x(\underline{0}) = x \quad ; \quad (d\exp_x)_{\underline{0}} = Id$$

- $\exists\, \rho(M)>0$ s.t. $\exp_x\left[\{\vec{v}\in T_xM : \|\vec{v}\|<r(M)\}\right] \supseteq B(x;\rho(M))$

- Decreasing $r(M)$, we may assume that $\exp_x : B(0;r(M)) \to$ image is bi-Lipschitz with Lipschitz constant of $\exp_x, \exp_x^{-1} \le 2$.

Def$^{\underline{n}}$. Fix $x\in\Lambda$. The Pesin chart at $x\in\Lambda$ is the map $\psi_x : \mathbb{R}^2 \to M$, $\psi_x(\underline{v}) = \exp_x[C_\chi(x)\underline{v}]$.

Fix $\chi>0$ s.t. $0<\lambda e^{\chi}<1$ (λ from def$^{\underline{n}}$ of Λ)

Fix $C_0>0$ as above.

Thm (Pesin): Suppose $T: M \to M$ is a C^2 diffeo* of a compact Riemannian manifold M, and let Λ be a hyperbolic set. For all $\varepsilon > 0$, $\exists\, q_0, \delta_0 > 0$ s.t. for all $x \in \Lambda$

(1) $\psi_x : \{ \underline{v} \in \mathbb{R}^2 : \|\underline{v}\| < q_0 \} \to U_x$ is a diffeomorphism onto an open neighborhood U_x of x which contains a ball of fixed size $B(x, \delta_0)$.

(2) The maps in (1) are uniformly bi-Lipschitz

(3) "T in coordinates" given by

$$F_x := \psi_{T(x)}^{-1} \circ T \circ \psi_x : \{ \underline{v} \in \mathbb{R}^2 : \|\underline{v}\| < q_0 \} \to \mathbb{R}^2$$

has the form

$$\boxed{F_x(\xi, \eta) = \left(A(x)\xi + h_1(\xi, \eta),\ B(x)\eta + h_2(\xi, \eta) \right)}$$

where:

- contraction in ξ: $\quad C_0^{-1} \leq |A(x)| < e^{-\chi}$
- expansion in η $\quad e^{\chi} \leq |B(x)| \leq C_0$
- $h_i(\xi, \eta)$ are "negligible": $h_i(0,0) = 0$, $(dh_i)_{\underline{0}} = 0$,

$$\|\underline{v}\| < q_0 \Rightarrow \|(dh_i)_{\underline{u}}\| \leq \varepsilon \|\underline{u}\|$$

ε appears here

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* Remark: C^2 can be relaxed to having Hölder continuous first derivative.

Sketch of Proof. The proof is done by examining the Taylor expansion of $F_x(\xi,\eta)$ at $(0,0)$, noting that

- F_x is continuously differentiable twice* (C^2-assumption)
- $F_x(0,0)=(0,0)$ (because $\exp^{-1}_{T(x)}(T(x))=\underline{0}$)
- $$(dF_x)_{\underline{0}} = d\left(C_x(Tx)^{-1}\circ \exp^{-1}_{Tx}\circ T\circ \exp_x\circ C_x(x)\right)_{\underline{0}} =$$
$$= C_x(Tx)^{-1}(d\exp^{-1}_{Tx})_{Tx}\, dT_x\, d(\exp_x)_{\underline{0}}\, C_x(x)$$
$$= C_x(Tx)^{-1}\, dT_x\, C_x(x) \quad (\because (d\exp_x)_{\underline{0}}=\mathrm{Id};\ (d\exp^{-1}_{Tx})_{Tx}=\mathrm{Id})$$
$$= \text{diagonal matrix} \begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix},\quad |A|<e^{-\chi},\ |B|>e^{\chi}.$$
- The error term (h_1,h_2) vanishes together with its 1^{st} derivatives at $\underline{0}$.

The size of the neighborhood of $\underline{0}$ in $\mathbb{R}^2$ where the error has small derivative is controlled by the size of the second derivatives of F_x at $\underline{0}$, and these are bounded, since

- T is C^2 on M and M is compact;
- $\|C_\chi(\cdot)\|$, $\|C_\chi(\cdot)^{-1}\|$ are uniformly bounded.

Since $\psi_x = \exp_x\circ C_\chi(x)$ is uniformly bi-Lipschitz on this neighborhood (again since $\|C_\chi(\cdot)\|$, $\|C_\chi(\cdot)^{-1}\|$ are globally bounded), it maps to a neigh of x in M with size bounded below (i.e. containing a ball $B(x,\delta_0)$ with fixed δ_0). $\square$

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* See the footnote in the previous page

Exponential Sensitivity to Initial Conditions (ESIC)

Theorem. Suppose Λ is a hyperbolic set for a C^2 diffeomorphism on a compact Riemannian manifold. Then T has exponential sensitivity to initial conditions on Λ:

$$\exists r>0,\ \chi>0\ \forall x\in\Lambda\ \forall n>0\ \exists y\in\Lambda \text{ s.t.}$$
$$d(y,x)\le e^{-\chi n} \text{ but } d(T^k x, T^k y)>r \text{ for some } 0\le k\le n.$$

Proof. Let λ, χ, C_0 be the constants from the Oseledets Pesin reduction. We fix $x\in\Lambda$, and work with the Pesin charts along the orbit of x, with parameter ε so small that

$$e^{\chi}-2\varepsilon > e^{\chi/2},\quad e^{-\chi}+2\varepsilon<1.$$

Recall that $\exists q_0, \delta_0$ s.t.

$$\psi_{T^k(x)} : \{ \underline{v}\in\mathbb{R}^2 : \|\underline{v}\|_{\mathbb{R}^2}<q_0\} \longrightarrow \text{neigh of } T^k(x) \text{ containing } B(T^k x, \delta_0).$$

Observe that

$$\psi^{-1}_{T^n(x)}\circ T^n\circ\psi_x =$$
$$=\left(\psi^{-1}_{T^n(x)}\circ T\circ\psi_{T^{n-1}(x)}\right)\circ\left(\psi^{-1}_{T^{n-1}(x)}\circ T\circ\psi_{T^{n-2}x}\right)\circ\cdots\circ\left(\psi^{-1}_{Tx}\circ T\circ\psi_x\right)$$
$$= F_{T^{n-1}x}\circ F_{T^{n-2}x_{-1}}\circ\cdots\circ F_x, \text{ where}$$

$$F_{T^k x} := \psi_{T^{k+1}x}\circ T\circ\psi_{T^k x} \text{ encodes}$$
$$T : \left(\text{neigh of } T^k x\right) \longrightarrow \left(\text{neigh of } T^{k+1}x\right).$$

We write for short

$$F_k(\xi,\eta) = F_{T^k x}(\xi,\eta) = \left(A_k\xi + h_k^1(\xi,\eta),\ B_k\eta + h_k^2(\xi,\eta)\right)$$

where

$$|A_k| < e^{-\chi}, \quad |B_k| > e^{\chi}, \quad \|(dh_k^i)_{\underline{u}}\| \lesssim \varepsilon \|\underline{u}\|^{1/2}$$

Fix some point $y = \Psi_x(0, \eta_0)$, with $(0, \eta_0) \in$ Pesin chart of x. Let N be the first time $T^N(y) \notin B(T^n x, \rho)$.

Then $T^n(y) \in$ Pesin chart of $T^n(x)$ for $n = 0, 1, 2, \ldots, N-1$.

Write $\quad T^n(y) = \Psi_{T^n(x)}(\xi_n, \eta_n)$, with $|\xi_n|, |\eta_n| = O(1)$.

We have the recursion formula $(\xi_{n+1}, \eta_{n+1}) = F_n(\xi_n, \eta_n)$, i.e.

$$\begin{cases} \xi_{n+1} = A_n \xi_n + h_n^1(\xi_n, \eta_n) \\ \eta_{n+1} = B_n \eta_n + h_n^2(\xi_n, \eta_n) \end{cases} \qquad (0 \le n \le N)$$

Note that $|A_n| \le e^{-\chi}$, $|B_n| \ge e^{\chi}$ and

$$|h_n^i(\xi_n, \eta_n)| = |h_n^i(\xi_n, \eta_n) - h_n^i(0,0)| \le \max \|\nabla h^i\| \cdot \left\| \binom{\xi_n}{\eta_n} \right\| \quad (\because h_n(0,0) = 0)$$

$$\le \varepsilon\left(|\xi_n| + |\eta_n|\right).$$

Thus

$$(*)\quad \begin{cases} |\xi_{n+1}| \le (e^{-\chi} + \varepsilon)|\xi_n| + \varepsilon|\eta_n| \\ |\eta_{n+1}| \ge (e^{\chi} - \varepsilon)|\eta_n| - \varepsilon|\xi_n| \end{cases} ; \quad \begin{matrix} \xi_0 = 0 \\ \eta_0 \ne 0 \end{matrix}$$

<u>Claim</u>. Suppose ε is so small that $(e^{-\chi}+2\varepsilon)<1$, $(e^{\chi}-2\varepsilon)>1$.
Then the solution to $(*)$ satisfies

$$|\eta_k| \ge (e^{\chi}-2\varepsilon)^k |\eta|_0 \qquad (k=0,1,\dots,N)$$

<u>Proof</u>. First we show that $|\xi_k| \le |\eta_k|$ and $|\eta_{k+1}| \ge |\eta_k|$.

- <u>$k=0$</u>: $|\xi_0| \le |\eta_0|$ because $\xi_0 = 0, \eta_0 \ne 0$.
  $|\eta_1| \ge |\eta_0|$ because of $(*)$.
- <u>Induction Step</u>: Assume $|\xi_k| \le |\eta_k|$ and $|\eta_{k+1}| \ge |\eta_k|$, then
  $$|\xi_{k+1}| \le (e^{-\chi}+\varepsilon)|\xi_k| + \varepsilon|\eta_k| \le (e^{-\chi}+2\varepsilon)|\eta_k| < |\eta_k| \le |\eta_{k+1}|$$
  $$|\eta_{k+1}| \ge (e^{\chi}-\varepsilon)|\eta_k| - \varepsilon|\eta_k| \ge (e^{\chi}-2\varepsilon)|\eta_k| > |\eta_k|$$
  as required.

Substituting these inequalities in $(*)$, we obtain

$$|\eta_{k+1}| \ge (e^{\chi}-\varepsilon)|\eta_k| - \varepsilon|\eta_k| = (e^{\chi}-2\varepsilon)|\eta_k|$$

The claim follows.

Recall that $N :=$ first n s.t. $T^N(y) \notin$ Pesin chart of $T^N(x)$
(or $N := +\infty$ if there is no such n.)

Let $y := \psi_x(\xi_0, \eta_0)$, then

- $T^N(y)$ is outside the Pesin chart of $T^n(x)$, whence $d(T^N y, T^N x) > \delta_0$
- $\forall\, 0 \le k \le N$ $d(T^k y, T^k x) = d\left(\exp_{T^k x}\left[C_x(T^k x)\begin{pmatrix}\xi_k\\ \eta_k\end{pmatrix}\right], \exp_{T^k x}\left[C_x(T^k x)\begin{pmatrix}0\\0\end{pmatrix}\right]\right)$

$$\ge \mathrm{Lip}\left(\exp^{-1}_{T^k x}\right)^{-1} \left\| C_x(T^k x)\begin{pmatrix}\xi_k\\ \eta_k\end{pmatrix}\right\|$$

$$\ge \mathrm{Lip}\left(\exp^{-1}_{T^k x}\right)^{-1} \cdot \left\| C_x(T^k x)^{-1}\right\|^{-1} \sqrt{\xi_k^2 + \eta_k^2}$$

$$\ge K^{-1}|\eta_k| \text{, where } K = \sup_M \left(\mathrm{Lip}\, \exp_y^{-1}\right) \cdot \sup_\Lambda \|C_x(\cdot)^{-1}\|$$

$$\ge K^{-1}\gamma^k|\eta_0| \text{, where } \gamma := (e^\chi - 2\varepsilon) > 1$$

- $$d(y,x) = d(\psi_x(\xi_0,\eta_0), \psi_x(0,0))$$

$$= d\left(\exp_x\left[C_x(x)\begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix}\right], \exp_x\left[C_x(x)\begin{pmatrix}0\\0\end{pmatrix}\right]\right)$$

$$\le \underbrace{\mathrm{Lip}(\exp_x) \cdot \|C_x(x)\|}_{\text{globally bounded}} \cdot \sqrt{\xi_0^2 + \eta_0^2} \le L|\eta_0|$$

for some global constant L.

We can now deduce ESIC at x:

Take $\eta_0 = \frac{1}{L}\gamma^{-n}$, then $d(y,x) < \gamma^{-n}$.

- Case 1 ($N \le n$): Take $k := N$, then $1 \le k \le n$ and
$$d(T^k y, T^k x) \equiv d(T^N y, T^N x) > \delta_0$$
- Case 2 ($N > n$): Take $k := n$, then $1 \le k \le n$ and
$$d(T^k y, T^k x) = d(T^N y, T^N x) \ge K^{-1}\gamma^n \cdot |\eta_0| = \frac{1}{KL}.$$

Taking $r_0 := \min\{\delta_0, 1/KL\}$, we see that in all cases, $d(y,x) \le \gamma^{-n}$, and $\exists\, 1 \le k \le n$ such that $d(T^k y, T^k x) \ge r_0$. We proved ESIC $\square$