Lecture 11: Structural Stability 1

The Stability Problem: Study the sensitivity of a model to

- errors in the initial condition ← last lecture
- errors in the model itself (i.e. the map)
- errors in each iteration (noise, numerical errors)

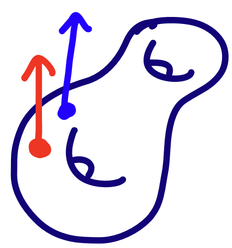
} remaining lectures

Preliminary Def$^{\underline{n}}$s

Let Diff(M) denote the collection of diffeomorphisms of a Riemannian manifold M. We'll use two notions of distance on M

- C^0-distance: $\text{dist}_{C^0}(T,S) := \sup_{x\in M} \text{dist}(T(x), S(x))$

- C^1-distance: Embedd M isometrically into $\mathbb{R}^N$, and view vectors in T_xM as vectors in $\mathbb{R}^N$.



$$\text{dist}_{C^1}(T,S) = \sup_{x\in M} \text{dist}(Tx, Sx) + \\ + \sup_{x\in M} \sup_{\substack{\vec{v}\in T_xM \\ \|\vec{v}\|=1}} \text{dist}(dT_x\vec{v}, dS_x\vec{v})$$

How to think about this: If $\text{dist}_{C^1}(T,S)$ is small, and we use a coordinate chart $(x_1, \dots, x_d)$ for M to express

$$T(x_1, \dots, x_d) = \begin{pmatrix} F_1(\underline{x}) \\ \vdots \\ F_d(\underline{x}) \end{pmatrix}, \quad S(x_1, \dots, x_d) = \begin{pmatrix} G_1(\underline{x}) \\ \vdots \\ G_d(\underline{x}) \end{pmatrix}$$

then $\|\vec{F}-\vec{G}\|_\infty$, $\left\| \frac{\partial(F_1, \dots, F_d)}{\partial(x_1, \dots, x_d)} - \frac{\partial(G_1, \dots, G_d)}{\partial(x_1, \dots, x_d)} \right\|$ are small.

To save time, we'll express all our stability results as consequences of one "master theorem" which we'll prove at the end of the course.

"Diagram Fixing Thm"*: Suppose $T: M \to M$ is a C^1 diffeomorphism with a hyperbolic set Λ. There exist

- $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$
- an open set $\mathcal{O} \supseteq \Lambda$, as follows.

Suppose S is a diffeo s.t. $\mathrm{dist}_{C^1}(T, S) < \varepsilon_1$,
$h: X \to X$ is a homeomorphism of a metric space X
and $\phi: X \to \mathcal{O}$ is a continuous map s.t. $\overline{\phi(X)} \subseteq \mathcal{O}$ and
$\mathrm{dist}_{C^0}(\phi \circ h, S \circ \phi) < \varepsilon_2$, i.e.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ \mathcal{O} \arrow[r, "S"'] & M \end{tikzcd}$$

"ε_2-commutes"

Then: $\exists!$ continuous $\psi: X \to \mathcal{O}$ s.t. $\mathrm{dist}_{C^0}(\psi, \phi) < \varepsilon_3$ and $\psi \circ h = S \circ \psi$, i.e.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ \mathcal{O} \arrow[r, "S"'] & M \end{tikzcd}$$

exactly commutes

Moreover: $\forall \varepsilon \, \exists \delta$ s.t.

$$\mathrm{dist}_{C^0}(\phi \circ h, S \circ \phi) < \delta \Rightarrow \mathrm{dist}_{C^0}(\psi, \phi) < \varepsilon.$$

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* this is not a standard name.

Application 1: Shadowing Theory

Def^n. A δ-pseudo orbit is a sequence of points $\{x_k\}$ s.t. $\mathrm{dist}(x_{k+1}, T(x_k)) < \delta$ for all k.

Example. If we calculate $T^k(x)$ numerically with truncation error $< \delta$, what we obtain is a δ-pseudo-orbit

Anosov Shadowing Lemma. Suppose Λ is a hyperbolic set of a C^1 diffeomorphism $T: M \to M$. Then $\forall \varepsilon\ \exists \delta$ s.t. for any δ-pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ inside Λ, $\exists x$ in M s.t.

$$d(T^k x, x_k) < \varepsilon \text{ for all } k \in \mathbb{Z}.$$

We say: "the orbit of x ε-shadows the orbit of (x_k)".

Proof. Consider the diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "h"] \arrow[d, "\phi"'] & \mathbb{Z} \arrow[d, "\phi"] \\ \Theta \arrow[r, "T"'] & \Theta \end{tikzcd} \qquad \begin{array}{l} h(z) = z+1 \\ \phi(k) = x_k \end{array}$$

It δ-commutes:

$$d(\phi\circ h, T\circ\phi) = \sup_k d((\phi\circ h)(k), (T\circ\phi)(k))$$
$$= \sup_k (x_{k+1}, T(x_k)) < \delta$$

By the diagram fixing theorem, if $\delta < \varepsilon_2$ then $\exists \psi: \mathbb{Z} \to \Theta$ s.t. $\psi\circ h = T\circ\psi$.

Moreover, given ε, we can chose δ s.t. $\mathrm{dist}_{C^0}(\psi, \phi) < \varepsilon$.

Let $x := \psi(0)$. Then

$$d(T^k(x), x_k) = d((T^k \circ \psi)(0), x_k) = d(\psi(h^k(0)), x_k)$$

$$= d(\psi(k), x_k) = d(\psi(k), \phi(k)) < \mathrm{dist}_{C^0}(\psi, \phi) < \varepsilon.$$

So the orbit of x shadows (x_k). $\square$

Application 2: Existence of Periodic Orbits

Anosov "Closing Lemma": Suppose Λ is a hyperbolic set of a C^1 diffeomorphism. For every ε $\exists \delta$ as follows. Suppose $x \in \Lambda$ and $d(T^N x, x) < \delta$. Then $\exists y$ s.t. $T^N(y) = y$ and $d(T^k y, T^k x) < \varepsilon$ $(k = 0, 1, .., N)$.

Proof. Apply the "diagram fixing theorem" to

$$\begin{tikzcd} \mathbb{Z}/N\mathbb{Z} \arrow[r, "h"] \arrow[d, "\phi"'] & \mathbb{Z}/N\mathbb{Z} \arrow[d, "\phi"] \\ \Theta \arrow[r, "T"'] & \Theta \end{tikzcd}$$

$h(k) = k+1 \pmod N$

$\phi(k) = T^k(x)$

The diagram δ-commutes, because

$$d((\phi \circ h)(k), (T \circ \phi)(k)) = d(T^{(k+1) \bmod N} x, T(T^{k \bmod N} x))$$

$$= \begin{cases} k = 0, 1, .., N-2: & = 0 \\ k = N-1 & = d(T^0 x, T^N x) < \delta. \end{cases}$$

If $\delta < \varepsilon_2$, $\exists \psi : \mathbb{Z}/N\mathbb{Z} \to \Theta$ s.t. $T \circ \psi = \phi \circ h$.

Moreover, if δ is sufficiently small, $\mathrm{dist}_{C^0}(\psi, \phi) < \varepsilon$.
Let $y := \psi(0)$. Then

$$d(y, x) = d(\psi(0), \phi(0)) < \varepsilon$$

$$T^N(y) = (T^N \circ \psi)(0) = (\psi \circ h^N)(0) = \psi(0) = y. \quad \square$$

<u>Thm</u>. Suppose Λ is a hyperbolic set. Then for every $\varepsilon > 0$, there is a periodic orbit in the ε-neighborhood of Λ.

[Later, we'll see that this periodic orbit must be hyperbolic.]

<u>Proof</u>. Λ is compact and invariant.

By Birkhoff's theorem, $\exists x \in \Lambda$ which is recurrent:

$$\exists n_k \to \infty \qquad T^{n_k}(x) \longrightarrow x.$$

By Anosov's closing lemma, $\exists$ periodic orbits converging to x. $\square$

Application 3: Structural Stability

Structural Stability of Hyperbolic Sets: Let $T: M \to M$ be a C^1 diffeomorphism with a hyperbolic set Λ. Then $\exists$ open neigh $O \supseteq \Lambda$ and $\varepsilon > 0$ as follows.
Any C^1 diffeomorphism $S: O \to S(O)$ such that

$$\mathrm{dist}_{C^1}(T, S) < \varepsilon$$

has a hyperbolic set $\Lambda' \subseteq O$, and $\exists$ homeomorphism $\varphi: \Lambda \to \Lambda'$ s.t.

$$\begin{tikzcd} \Lambda \arrow[r, "T"] \arrow[d, "\varphi"'] & \Lambda \arrow[d, "\varphi"] \\ \Lambda' \arrow[r, "\varphi"'] & \Lambda' \end{tikzcd}$$

commutes.

Moreover, $\forall \delta\ \exists \varepsilon$ s.t. if $\mathrm{dist}_{C^1}(T, S) < \varepsilon$, then $\mathrm{dist}_{C^0}(\varphi, \mathrm{id}) < \delta$.

Remark 1: φ is called a topological conjugacy of $T|_{\Lambda}$, $S|_{\Lambda'}$

Remark 2: In general, φ is not differentiable.

Proof. Apply the diagram fixing theorem to
$$\begin{tikzcd} \Lambda \arrow[r, "T"] \arrow[d, "\mathrm{id}"'] & \Lambda \arrow[d, "\mathrm{id}"] \\ O \arrow[r, "S"'] & M \end{tikzcd}$$

If $\varepsilon < \varepsilon_2$, then this gives us $\varphi: \Lambda \to O$ continuous s.t.

$$\begin{tikzcd} \Lambda \arrow[r, "T"] \arrow[d, "\varphi"'] & \Lambda \arrow[d, "\varphi"] \\ O \arrow[r, "S"'] & O \end{tikzcd}$$

exactly commutes.

Let $\Lambda' := \varphi(\Lambda)$, then
$$\begin{tikzcd} \Lambda \arrow[r, "T"] \arrow[d, "\varphi"'] & \Lambda \arrow[d, "\varphi"] \\ \Lambda' \arrow[r, "S"'] & \Lambda' \end{tikzcd}$$
exactly commutes.

<u>Claim</u>: If ε is small enough, then φ is a homeomorphism.

<u>Proof</u>. We already know that φ is continuous. We need to show that it has a continuous <u>inverse</u>.

Fix ε' to be determined later, and assume $\mathrm{dist}_{C^1}(T,S)$ is so small that $\mathrm{dist}_{C^0}(\varphi, id) < \varepsilon'$.

Apply the diagram fixing theorem to

$$\begin{tikzcd} \Lambda' \arrow[r, "S"] \arrow[d, "id"'] & \Lambda' \arrow[d, "id"] \\ \mathcal{O} \arrow[r, "T"'] & M \end{tikzcd}$$

This gives $\tilde{\varphi} : \Lambda' \to \mathcal{O}$ continuous s.t. $\tilde{\varphi} \circ S = T \circ \tilde{\varphi}$.
In addition, if $\mathrm{dist}_{C^1}(S,T)$ is sufficiently small, $\mathrm{dist}_{C^0}(\tilde{\varphi}, id) < \varepsilon'$.

We obtain the following <u>commuting diagram</u>:

$$\begin{tikzcd} \Lambda \arrow[r, "T"] \arrow[d, "\varphi"'] & \Lambda \arrow[d, "\varphi"] \\ \Lambda' \arrow[r, "S"] \arrow[d, "\tilde{\varphi}"'] & \Lambda' \arrow[d, "\tilde{\varphi}"] \\ \mathcal{O} \arrow[r, "T"'] & M \end{tikzcd}$$

Observe that $\mathrm{dist}_{C^0}(\tilde{\varphi} \circ \varphi, id) < 2\varepsilon'$, because for all x

$$d(\tilde{\varphi}(\varphi(x)), x) \leq \mathrm{dist}(\tilde{\varphi}(\varphi(x)), \varphi(x)) + \mathrm{dist}(\varphi(x), x) < 2\varepsilon'.$$

We see that we have <u>two</u> commuting diagrams:

$$\begin{tikzcd} \Lambda \arrow[r, "T"] \arrow[d, "\tilde{\varphi}\circ\varphi"'] & \Lambda \arrow[d, "\tilde{\varphi}\circ\varphi"] \\ \mathcal{O} \arrow[r, "T"'] & M \end{tikzcd} , \qquad \begin{tikzcd} \Lambda \arrow[r, "T"] \arrow[d, "id"'] & \Lambda \arrow[d, "id"] \\ \mathcal{O} \arrow[r, "T"'] & M \end{tikzcd} .$$

If $\varepsilon' < \varepsilon_3 :=$ local uniqueness constant in the diagram fixing theorem, then necessarily $\tilde{\varphi}\circ\varphi = id$, and so $\varphi : \Lambda \to \varphi(\Lambda)$ has a continuous inverse.

<u>In summary</u>: $\exists$ compact S-invariant set $\Lambda' := \varphi(\Lambda)$ s.t. $T|_\Lambda$, $S|_\Lambda$ are topologically conjugate:

$$\begin{tikzcd} \Lambda \arrow[r, "T"] \arrow[d, "\varphi"'] & \Lambda \arrow[d, "\varphi"] \\ \Lambda' \arrow[r, "S"'] & \Lambda \end{tikzcd} \quad \text{commutes}, \quad \varphi = \text{homeomorphism}.$$

It remains to show that Λ' is a hyperbolic set for S. This requires further tools.