Lecture 12: Stability of Hyperbolic Sets 2

We continue with the proof of structural stability of hyperbolic sets:

Thm. Suppose Λ is a hyperbolic set of a C^1 diffeomorphism T, then $\exists$ open set $\mathcal{O} \supseteq \Lambda$ and $\exists\, \varepsilon > 0$ s.t. for every C^1 diffeo S s.t. $\mathrm{dist}_{C^1}(T, S) < \varepsilon$, $\exists$ compact invariant* $\Lambda' \subseteq \mathcal{O}$ s.t.

(1) $T|_{\Lambda}$ is topologically conjugate to $S|_{\Lambda'}$

(2) Λ' is a hyperbolic set of S.

We already saw (1). Today we'll prove (2), by showing the following stability result:

Thm: Suppose T is a C^1-diffeomorphism with a hyperbolic set Λ. There is an open set $U \supseteq \Lambda$ and $\varepsilon > 0$ as follows.
If $S : U \to M$ is a C^1-diffeomorphism s.t.

(a) $\mathrm{dist}_{C^1}(T, S) < \varepsilon$

(b) $\tilde{\Lambda} \subseteq U$ is a compact invariant set <u>for S</u>

Then $\tilde{\Lambda}$ is a hyperbolic set for S.

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* A set Λ' is T-invariant if $T^{-1}(\Lambda') = \Lambda'$. Necessarily (since T is invertible), $\forall x \in \Lambda'$ $T(x) \in \Lambda'$, $T^{-1}(x) \in \Lambda'$.

Tool 1: The Lyapunov Metric

Suppose Λ is a hyperbolic set.

Recall the Oseledets-Pesin reduction $C_\chi(x): \mathbb{R}^2 \to T_xM$:

$$C_\chi(Tx)^{-1}(dT)_x\, C_\chi(x) = \begin{pmatrix} A_x & 0 \\ 0 & B_x \end{pmatrix}, \quad \begin{array}{l} |A_x| < e^{-\chi} < 1 \\ |B_x| > e^{\chi} > 1 \end{array}$$

The **Lyapunov (Riemannian) metric** on Λ is the continuous family of inner products on T_xM $(x \in \Lambda)$ given by

$$\langle \vec{v}, \vec{w} \rangle'_x := \left(C_\chi(x)^{-1}\vec{v},\ C_\chi(x)^{-1}\vec{w} \right)_{\mathbb{R}^2} \qquad (\vec{v}, \vec{w} \in T_xM)$$

__Exercise__:

- $\langle \cdot, \cdot \rangle'_x$ is continuous on Λ
- $E^u(x) \perp E^s(x)$ with respect to $\langle \cdot, \cdot \rangle'_x$
- $\forall \vec{v} \in E^s(x), \forall n \geq 0 \quad \|(dT^n)_x \vec{v}\|'_x \leq e^{-n\chi} \|\vec{v}\|'_x$
- $\forall \vec{v} \in E^u(x), \forall n \geq 0 \quad \|(dT^{-n})_x \vec{v}\|'_x \leq e^{-n\chi} \|\vec{v}\|'_x$

$e^{-n\chi}\|\vec{v}\|'_x$ not $Ce^{-n\chi}\|\vec{v}\|_x$!

__Lemma__: $\exists$ global constant K_0 s.t. for all $x \in \Lambda$,

$$K_0^{-1} \|\cdot\|_x \leq \|\cdot\|'_x \leq K_0 \|\cdot\|_x$$

__Proof__. Suppose $\vec{v} \in T_xM$, $\vec{v} = C_\chi(x)\underline{v}$, $\underline{v} \in \mathbb{R}_d$.

Then
$$\|\vec{v}\|'_x = \|C_\chi(x)^{-1}\vec{v}\|_{\mathbb{R}^2} \leq \|C_\chi(x)^{-1}\|\, \|\vec{v}\|_x$$
$$\|\vec{v}\|'_x = \|C_\chi(x)^{-1}\vec{v}\|_{\mathbb{R}^2} \geq \|C_\chi(x)\|^{-1} \cdot \|\vec{v}\|_x .$$

The lemma follows with $K_0 := \max_{x \in \Lambda} \left(\|C_\chi(x)^{-1}\| + \|C_\chi(x)\|^{-1} \right)$ (which is finite by the continuity of $C_\chi(x)$ on Λ). $\square$

Tool 2: Continuous Extensions

Lemma 1: Let Λ be a compact subset of a Riemannian manifold.

(1) Any continuous real function on Λ has a continuous extension to an open neighborhood of Λ

(2) Any continuous vector field on Λ has a continuous extension to an open neighborhood of Λ

(3) Any continuous family of independent bases of T_xM $(x \in \Lambda)$ has a continuous extension to an open neighborhood of Λ

(4) Any continuous family of inner products on T_xM $(x \in \Lambda)$ has a continuous extension to an open neighborhood of Λ

Exercise:

(1) Look up "Tietze's Extension Theorem". Prove (1).

(2) Look up "partitions of unity". Prove (2).

(3) Prove (2) under the assumption that the manifold is embedded in euclidean space.

(4) Prove (3)

(5) Prove (4). (Hint: To specify an inner product, it's sufficient to specify an orthonormal basis)

Tool 3: The Cone Criterion

Setup:

- $T: M \to M$ C^1 diffeomorphism, $U \subseteq M$ open set
- $\|\cdot\|'_x$ is some continuous Riemannian norm on U s.t. $K_0^{-1} \leq \dfrac{\|\cdot\|'_x}{\|\cdot\|_x} \leq K_0$
  ($\|\cdot\|_x :=$ Riemannian norm on M).
- $X \subseteq M$ a compact invariant set.

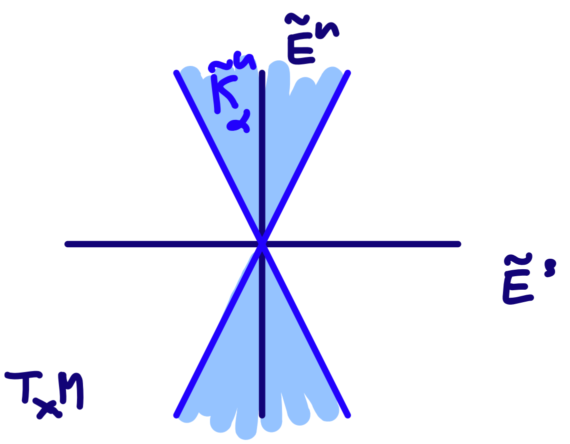
Terminology: Suppose $T_xM = \tilde{E}^u(x) \oplus \tilde{E}^s(x)$ ($x \in X$) is some continuous splitting (possibly non-invariant).

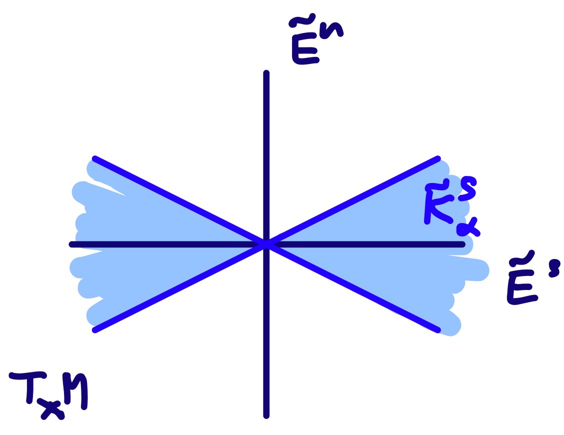
- Given $\vec{v} \in T_xM$, we split $\vec{v} = \vec{v}^u + \vec{v}^s$ ($\vec{v}^u \in \tilde{E}^u(x)$, $\vec{v}^s \in \tilde{E}^s(x)$)
- Given $\alpha \in (0,1)$, we define the cone families:

$$\tilde{K}^u_\alpha(x) := \{ \vec{v} \in T_xM : \|\vec{v}^s\|'_x \leq \alpha \|\vec{v}^u\|'_x \}$$

$$\tilde{K}^s_\alpha(x) := \{ \vec{v} \in T_xM : \|\vec{v}^u\|'_x \leq \alpha \|\vec{v}^s\|'_x \}$$

($x \in X$)





Cone Criterion (on Λ'): $\exists\, 0<\alpha<1$, $0<\mu<1$ and $\exists$ continuous splitting
$T_x M = \vec{E}^u(x) \oplus \vec{E}^s(x)$ $(x\in\Lambda')$ s.t. the associated α-cones satisfy:

(1) Almost Invariance: For all $x\in\Lambda'$

$$(dT)_x\, \tilde{K}^u_\alpha(x)\setminus\{0\} \subseteq \operatorname{int} \tilde{K}^u_\alpha(T(x)), \quad (dT^{-1})_x\, \tilde{K}^s_\alpha(x)\setminus\{0\} \subseteq \operatorname{int} \tilde{K}^s_\alpha(T^{-1}(x))$$

(2) Contraction Estimates:

$$\forall x\in\Lambda' \quad \begin{array}{ll} \forall \vec{v}\in \tilde{K}^s_\alpha(x)\setminus\{0\} & \|(dT)_x \vec{v}\|'_{Tx} < \mu \|\vec{v}\|'_x \\ \forall \vec{v}\in \tilde{K}^u_\alpha(x)\setminus\{0\} & \|(dT^{-1})_x \vec{v}\|'_{T^{-1}x} < \mu \|\vec{v}\|'_x \end{array}$$

Cone Lemma: Any compact invariant set Λ' satisfying the cone criterion is a hyperbolic set

Proof. We need to build the invariant splitting of $T_x M$.

Let
$$E^u(x) := \bigcap_{n=0}^{\infty} (dT^n)_{T^{-n}x}\, \tilde{K}^u_\alpha(T^{-n}(x))$$

(we'll have to prove that these are vector spaces)

$$E^s(x) := \bigcap_{n=0}^{\infty} (dT^{-n})_{T^n x}\, \tilde{K}^s_\alpha(T^n(x))$$

Claim 1: $(dT)_x(E^u(x)) = E^u(Tx)$, $(dT)_x(E^s(x)) = E^s(Tx)$

Proof.
$$(dT)_x E^u(x) = \bigcap_{n=0}^{\infty} (dT^{n+1})_{T^{-n}x}\, \tilde{K}^u_\alpha(T^{-n}x)$$
$$= \bigcap_{n=0}^{\infty} (dT^{n+1})_{T^{-(n+1)}Tx}\, \tilde{K}^u_\alpha(T^{-(n+1)}Tx)$$
$$= \bigcap_{\ell=1}^{\infty} \underbrace{(dT^{\ell})_{T^{-\ell}Tx}\, \tilde{K}^u_\alpha(T^{-\ell}Tx)}_{\subseteq \tilde{K}^u_\alpha(Tx) \text{ by almost invariance of } \tilde{K}^u_\alpha(\cdot)}$$
$$= \bigcap_{\ell=0}^{\infty} (dT^{\ell})_{T^{-\ell}Tx}\, \tilde{K}^u_\alpha(T^{-\ell}Tx) = E^u(Tx)$$

Similarly, $(dT^{-1})_x E^s(x) = E^s(T^{-1}x)$.

Claim 2: $\forall \vec{v} \in E^s(x)\ \ \forall n \geq 0,\ \ \|(dT^n)_x \vec{v}\|_{T^n x} \leq K_0^2 \mu^n \|\vec{v}\|_x$

$\forall \vec{v} \in E^u(x)\ \ \forall n \geq 0,\ \ \|(dT^{-n})_x \vec{v}\|_{T^{-n} x} \leq K_0^2 \mu^n \|\vec{v}\|_x$

$\forall \vec{v} \in E^u(x),\ \ \forall n \geq 0,\ \ \|(dT^n)_x \vec{v}\|_{T^n x} \geq K_0^{-2} \mu^{-n} \|\vec{v}\|_x$

Proof. If $\vec{v} \in E^s(x)$, then by the previous claim,

$$dT_x^k \vec{v} \in E^s(T^k x) \subseteq \tilde{K}_\alpha^s(T^k x) \quad \text{for all } k$$

By the forward contraction on $\tilde{K}_\alpha^s(\cdot)$,

$$\|dT_x^n \vec{v}\|' = \|dT_{T^{n-1}x}(\underbrace{dT_x^{n-1}\vec{v}}_{\in E^s(T^{n-1}x) \subseteq \tilde{K}_\alpha^s(T^{n-1}x)})\|' \leq \mu \|dT_x^{n-1}\vec{v}\|'$$

$$= \mu \|dT_{T^{n-2}x} \underbrace{dT_x^{n-2}\vec{v}}_{\in E^s(T^{n-2}x) \subseteq \tilde{K}_\alpha^s(T^{n-2}x)}\|' \leq \mu^2 \|dT_x^{n-2}\vec{v}\|'$$

$$\leq \cdots \leq \mu^n \|\vec{v}\|'$$

Since $K_0^{-1}\|\cdot\|_x \leq \|\cdot\|_x' \leq K_0 \|\cdot\|_x$,

$\forall n \geq 0 \quad \|(dT^n)_x \vec{v}\|_x \leq K_0^2 \mu^n \|\vec{v}\|_x$, on $E^s(x)$.

Similarly, $\forall n \geq 0$, $\|(dT^n)_x \vec{v}\|_x \geq K_0^{-2} \mu^{-n} \|\vec{v}\|_x$ on $E^u(x)$.

Substituting $dT^{-n}\vec{v}$ for $\vec{v}$, we get

$$\|\vec{v}\| = \|dT^n dT^{-n} \vec{v}\| \geq K_0^{-2} \mu^{-n} \|dT^{-n}\vec{v}\|,$$

whence $\|dT^{-n}\vec{v}\| \leq K_0^2 \mu^n \|\vec{v}\|$.

<u>Claim 3</u>. $E^u(x)$, $E^s(x)$ <u>contain</u> linear vector spaces of dimensions $\dim \tilde{E}^u(x)$, $\dim \tilde{E}^s(x)$, respectively.

<u>Proof</u> Fix <u>orthonormal</u> bases $\vec{w}_1^{(n)}, \dots, \vec{w}_p^{(n)}$ for $(dT^n)_x \tilde{E}^u(T^{-n}x)$.

Find a subsequence $n_k \to \infty$ s.t.

$$\vec{w}_i^{(n_k)} \xrightarrow[k\to\infty]{} \vec{w}_i \qquad (i=1,\dots,p)$$

The limit is an orthonormal system. Observe:

$$\forall n \geq N \quad \vec{w}_1^{(n)}, \dots, \vec{w}_p^{(n)} \in (dT^n)_x \tilde{K}_\alpha^u(T^{-n}x) \subseteq \underbrace{(dT^N)_x \tilde{K}_\alpha^u(T^{-N}x)}_{\text{closed set}}$$

Thus $\quad \vec{w}_1, \dots, \vec{w}_p \in (dT^N)_x \tilde{K}_\alpha^u(T^{-N}x)$ for all $N \geq 0$.

Thus $\quad \vec{w}_1, \dots, \vec{w}_p \in \bigcap_{N=0}^{\infty} (dT^N)_x \tilde{K}_\alpha^u(T^{-N}x) = E^u(x)$.

Next fix some $a_1, \dots, a_p \in \mathbb{R}$. Since $\tilde{E}^u(\cdot)$ are vector spaces,

$$\forall n \geq N \quad \sum_{i=1}^p a_i \vec{w}_i^{(n)} \in (dT^n)_x \tilde{E}^u(T^{-n}x) \subseteq (dT^n)_x \tilde{K}^u(T^{-n}x) \subseteq (dT^N)_x \tilde{K}^u(T^{-N}x)$$

Again, this leads to

$$\sum_{i=1}^p a_i \vec{w}_i \in \bigcap_{N=1}^{\infty} (dT^N)_x \tilde{K}^u(T^{-N}x) = E^u(x).$$

We see that $A^u := \mathrm{Span}\{w_1, \dots, w_p\}$ is inside $E^u(x)$.

By construction, A^u is a vector space of dimension $\dim \tilde{E}^u(x)$.

Similarly one can build a linear vector space $A^s \subseteq E^s(x)$ of dimension $\dim \tilde{E}^s(x)$.

Notice that $A^u \cap A^s \subseteq \tilde{E}^u \cap \tilde{E}^s = \{0\}$, $\dim A^s + \dim A^u = \dim \tilde{E}^u + \dim \tilde{E}^s = \dim M$.

So $T_x M = A^u \oplus A^s$.

Claim 4. $E^u(x)$, $E^s(x)$ are linear vector spaces, and $T_xM = E^u(x) \oplus E^s(x)$.

Proof. We claim that $E^s(x) = A^s(x)$. Indeed, pick some $\vec{v} \in E^s(x)$ and decompose $\vec{v} = \vec{a}^u + \vec{a}^s$ where $\vec{a}^t \in A^t$ $(t = u,s)$.

If $\vec{a}^u \neq 0$, then

$$\|(dT^n)_x \vec{v}\| \geq \|(dT^n)_x \vec{a}^u\| - \|(dT^n)_x \vec{a}^s\|$$

$A^u \subseteq E^u$, $A^s \subseteq E^s$

$$\geq K_0^{-2}\mu^{-n}\|\vec{a}^u\| - K_0^2 \mu^n \|\vec{a}^s\| \xrightarrow[n\to\infty]{} \infty, \text{ unless } \vec{a}^u = 0.$$

But $\|(dT^n)_x \vec{v}\| \leq K_0^2 \mu^n \|\vec{v}\| \longrightarrow 0$, because $\vec{v} \in E^s$.
Necessarily $\vec{a}^u = 0$. This shows:

$$E^s(x) \subseteq A^s(x) \Longrightarrow E^s(x) = A^s(x)$$

Similarly $$E^u(x) \subseteq A^u(x) \Longrightarrow E^u(x) = A^u(x).$$

Thus $E^u(x)$, $E^s(x)$ are linear spaces.

Observe that

$$E^u(x) \cap E^s(x) = A^u(x) \cap A^s(x) \subseteq \tilde{E}^u(x) \cap \tilde{E}^s(x) = \{0\}.$$

Since $$\dim E^u + \dim E^s = \dim A^u + \dim A^s = \dim \tilde{E}^u + \dim \tilde{E}^s = \dim T_xM,$$

we have $T_xM = E^u(x) \oplus E^s(x)$.

By claims 1,2 and 4, X is a hyperbolic set. $\square$

Stability of Hyperbolic Sets

Thm: Suppose T is a C^1-diffeomorphism with a hyperbolic set Λ. There is an open set $U \supseteq \Lambda$ and $\varepsilon > 0$ as follows. If $S: U \to M$ is a C^1-diffeomorphism s.t.

(a) $\mathrm{dist}_{C^1}(T, S) < \varepsilon$

(b) $X \subseteq U$ is a compact invariant set for S

then X is a hyperbolic set for S.

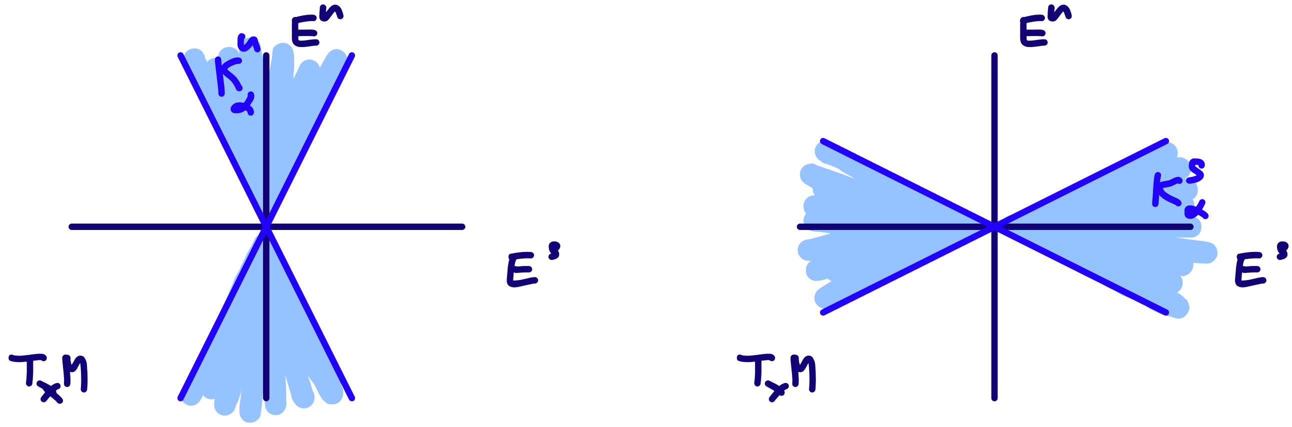
Lemma 1: T satisfies the cone criterion on Λ.

Proof: Let $T_x M = E^u(x) \oplus E^s(x)$ $(x \in \Lambda)$ be the T-invariant splitting associated with the hyperbolic set Λ. Let $\|\cdot\|_x'$ be the Lyapunov metric, and fix $0 < \eta < 1$ s.t. $\|dT|_{E^s}\| < \eta$, $\|dT^{-1}|_{E^u}\| < \eta$.

Fix $0 < \alpha < 1$. The α-(un)stable cone family is

$$K_\alpha^u(x) := \{ \vec{v} \in T_x M : \|\vec{v}^s\|_x' \leq \alpha \|\vec{v}^u\|_x' \}$$

$$K_\alpha^s(x) := \{ \vec{v} \in T_x M : \|\vec{v}^u\|_x' \leq \alpha \|\vec{v}^s\|_x' \}$$



Think of $K_\alpha^u(x)$, $K_\alpha^s(x)$ as of neighborhoods of $E^u(x)$, $E^s(x)$.

Claim: For all $0<\alpha<1$ sufficiently small, for all $x\in\Lambda$

(a) $dT_x(K^u_\alpha(x)\setminus\{0\})\subseteq \operatorname{int} K^u_\alpha(Tx)$

(b) $(dT_x)^{-1}(K^s_\alpha(x)\setminus\{0\})\subseteq \operatorname{int} K^s_\alpha(T^{-1}x)$

Proof: Suppose $v\in K^u_\alpha(x)\setminus\{0\}$, and split $v=v^u+v^s$, $v^t\in E^t(x)$.
Then $(dT)_x v=(dT)_x v^u+(dT)_x v^s$, $dT_x v^t\in E^t(Tx)$, and

$$\frac{\|dT_x v^s\|}{\|dT_x v^u\|}\leq \frac{\eta\|v^s\|}{\eta^{-1}\|v^u\|}=\eta^2\frac{\|v^s\|}{\|v^u\|}\leq \eta^2\alpha<\alpha$$

so $dT_x v\in \operatorname{int} K^u_\alpha(Tx)$. This proves (a). Similarly, one proves (b).

Claim: For all $0<\alpha<1$ sufficiently small, $\exists\mu\in(0,1)$ s.t. for all $x\in\Lambda$

(a) $\forall v\in K^u_\alpha(x)$, $\|dT^{-1}_x v\|'\leq\mu\|v\|'$

(b) $\forall v\in K^s_\alpha(x)$, $\|dT_x v\|'\leq\mu\|v\|'$

Proof. Fix $v\in K^u_\alpha(x)$ and split $v=v^u+v^s$, $v^t\in E^t(x)$.
Recall that $E^u\perp E^s$ with respect to $\|\cdot\|'$. So

$$\|dT^{-1}_x v\|'=\|dT^{-1}_x v^u+dT^{-1}_x v^s\|'=\sqrt{\|dT^{-1}_x v^u\|'^2+\|dT^{-1}_x v^s\|'^2}$$
$$\leq\sqrt{\eta^{-2}\|v^u\|'^2+M^2\|v^s\|'^2},\quad \text{where } M=\max_x \max\|dT^{-1}_x\|'$$
$$\leq\sqrt{\eta^{-2}\|v^u\|'^2+M^2\alpha^2\|v^u\|'^2}\qquad (\because\ \|v^s\|'\leq\alpha\|v^u\|' \text{ on } K^u_\alpha)$$
$$\leq\sqrt{\eta^{-2}+M^2\alpha^2}\cdot\|v^u\|'\leq\sqrt{\eta^{-2}+M^2\alpha^2}\,\|v\|'.$$

For α small enough, $\sqrt{\eta^{-2}+M^2\alpha^2}<\mu<1$, and we get (a).
(b) has a similar proof. The lemma follows. □

Lemma 2: Fix α as in Lemma 1, and let $\tilde{E}^u, \tilde{E}^s, \|\cdot\|'$ be continuous extensions of $E^u(x), E^s(x)$, and Lyapunov's metric to some open neighborhood U of Λ. Then $\exists\, \varepsilon>0$ and an open set $\mathcal{O} \supseteq \Lambda$ s.t. $\mathcal{O} \subseteq U$ and for every C^1 diffeo S s.t. $\mathrm{dist}_{C^1}(T,S)<\varepsilon$, S satisfies the cone criterion on $\mathcal{O}$ with the α-cones associated to $\tilde{E}^u, \tilde{E}^s$.

Proof. Let $\pi^u_x : T_xM \to \tilde{E}^u(x)$, $\pi^s_x : T_xM \to \tilde{E}^s(x)$ $(x \in U)$ be projections associated to the splitting $T_xM = \tilde{E}^u(x) \oplus \tilde{E}^s(x)$. Since $E^t(\cdot)$ are continuous, $x \mapsto \pi^{u,s}_x$ are continuous (on U).

For $x \in \Lambda$, $(dT_x)(K^u_\alpha(x)\setminus\{0\}) \subseteq \mathrm{int}\, K^u_\alpha(Tx)$ and $\|dT_x^{-1} v\|' \leq \eta \|v\|'$ for $v \in K^u_\alpha(x)$. Equivalently

$$\left.\begin{aligned} &\frac{\|(\pi^s_{T(x)} \circ dT_x)(v)\|}{\|(\pi^u_{T(x)} \circ dT_x)(v)\|} < \alpha \\ &\|dT_x^{-1} v\| < \eta \end{aligned}\right\} \text{ for all } x \in \Lambda \text{ and any unit vector } v \in K^u_\alpha(x).$$

The set $\{(x,w) : x \in \Lambda,\, w \in K^u_\alpha(x),\, \|w\| = 1\}$ is compact and $y \mapsto \pi^u_y, \pi^s_y, dT_y$ are continuous on U. A compactness argument thus shows that for some $\delta > 0$,

$$\left.\begin{aligned} &\frac{\|(\pi^s_{T(x)} \circ dT_x)(v)\|}{\|(\pi^u_{T(x)} \circ dT_x)(v)\|} < \alpha - \delta \\ &\|dT_x^{-1} v\| < \eta - \delta \end{aligned}\right\} \text{ for all } x \in \Lambda \text{ and any unit vector } v \in K^u_\alpha(x).$$

By the continuity of $\pi^{s,u}_y$, dT_y in y, $\exists$ open neigh $\mathcal{O}_1 \supseteq \Lambda$ s.t.

$$\left.\begin{aligned} &\frac{\|(\pi^s_{T(x)} \circ dT_x)(v)\|}{\|(\pi^u_{T(x)} \circ dT_x)(v)\|} < \alpha - \frac{\delta}{2} \\ &\|dT_x^{-1} v\| < \eta - \delta/2 \end{aligned}\right\} \quad \text{for all } x \in \mathcal{O}_1 \text{ and any unit vector } v \in \tilde{K}^u_\alpha(x).$$

Now fix some $\varepsilon > 0$ so small that for all S s.t.

$$\mathrm{dist}_{C^1}(T, S) < \varepsilon_1,$$

$$\left.\begin{aligned} &\frac{\|(\pi^s_{S(x)} \circ dS_x)(v)\|}{\|(\pi^u_{S(x)} \circ dS_x)(v)\|} < \alpha \\ &\|dS_x^{-1} v\| < \eta \end{aligned}\right\} \quad \text{for all } x \in \mathcal{O}_1 \text{ and any unit vector } v \in \tilde{K}^u_\alpha(x).$$

For such ε_1, $dS_x(\tilde{K}^u_\alpha(x) \setminus \{0\}) \subseteq \mathrm{int}\, \tilde{K}^u_\alpha(Sx)$ $\forall x \in \mathcal{O}_1$, and $\|dS_x^{-1}\| \leq \eta$ on $\tilde{K}^u_\alpha(x)$.

Similarly $\exists\, \varepsilon_2$ and $\mathcal{O}_2 \supseteq \Lambda$ open s.t. $dS_x(\tilde{K}^s_\alpha(x) \setminus \{0\}) \subseteq \mathrm{int}\, \tilde{K}^s_\alpha(Sx)$ $\forall x \in \mathcal{O}_2$, and $\|dS_x\| \leq \eta$ on $\tilde{K}^s_\alpha(x)$.

The lemma follows with $\mathcal{O} := \mathcal{O}_1 \cap \mathcal{O}_2$, $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. $\square$

Completion of the Proof of the Structural Stability Theorem:

Fix $\mathcal{O}, \varepsilon$ as in the previous lemma.
Fix $\delta>0$ so small that the δ-neigh of Λ is inside $\mathcal{O}$.
We saw in the previous lecture that $\exists\, U \supseteq \Lambda$ open and $\exists\, \varepsilon' \in (0,\varepsilon)$ s.t. if $\mathrm{dist}_{C^1}(T,S)<\varepsilon'$, then:

S has a compact invariant set Λ' with a homeomorphism $\varphi: \Lambda \to \Lambda'$ s.t. $\mathrm{dist}_{C^0}(\varphi, id)<\delta$, and

$$\begin{array}{ccc} \Lambda & \xrightarrow{T} & \Lambda \\ \varphi \downarrow & & \downarrow \varphi \\ \Lambda' & \xrightarrow[S]{} & \Lambda' \end{array} \quad \text{commutes.}$$

Since $\mathrm{dist}_{C^0}(\varphi, id)<\delta$, $\Lambda' = \varphi(\Lambda) \subseteq \delta\text{-neigh of } \Lambda \subseteq \mathcal{O}$.

By the previous lemma, S satisfies the cone condition on Λ.
It follows that Λ' is a hyperbolic set for S. $\square$

In summary: The hyperbolicity mechanism for exponential sensitivity to initial conditions is stable