Lecture 3: Dense Orbits

Last Time:

- Defined orbits of dynamical systems: $\{T^n(x)\}_{n=0}^{\infty}$, $T^n = \underbrace{T \circ \cdots \circ T}_{n}$
- Saw examples of complicated orbits:
  * large orbit closure / ω-limit set
  * "anything could happen" irregularities
  * exponential sensitivity to initial conditions

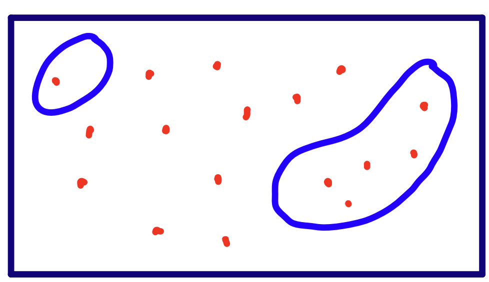
These examples were constructed by explicit, but example-specific tools.

"Modern" Theory of Dynamical Systems: General, non example-specific, tools for
  * proving the existence of (many) complicated orbits
  * sometimes, describing their behavior.

Today: Existence of dense orbits.

Def$^{\underline{n}}$. Suppose (X, d) is a metric space. A set $D \subseteq X$ is called dense, if it intersects every non-empty open set

Example: $\mathbb{Q} \subseteq \mathbb{R}$



Minimality

Let $T: X \to X$ be a continuous map on a metric space X.

Def$^{\underline{n}}$. A minimal set is a non-empty closed forward-invariant set ($T(A) \subseteq A$), without proper subsets with these properties.

(In lecture 1, we saw that minimal sets always exist when X is compact.)

Def$^{\underline{n}}$. A continuous map $T: X \to X$ is called minimal if X itself is a minimal set.

Thm. T is minimal iff $\forall x \in X$, $\overline{\{T^n(x) : n \geq 0\}} = X$.

Proof: ($\Rightarrow$) Suppose T is minimal, and $x \in X$. The set $A = \overline{\{T^n(x) : n \geq 0\}}$ is non-empty, closed, and forward invariant (exercise). So it equals X.

Fix some open set $U \neq \phi$, and choose $z \in U \cap A$. Since $z \in A$, $\exists\, n_k \geq 1$ s.t. $T^{n_k}(x) \to z$. Since U is open, $T^{n_k}(x) \in U$ for all k large enough. So $\{T^n(x)\}_{n \geq 0}$ intersects any open set, and we proved that x has a dense orbit.

($\Leftarrow$) If A is minimal and $x \in A$, then $B = \overline{\{T^n(x) : n \geq 0\}}$ is a closed non empty forward subset of A, so $B = A$. $\square$

Example: Irrational translations

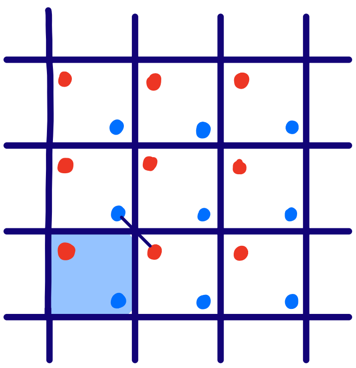
Torus: $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d = [0,1)^d$ with addition mod $\mathbb{Z}^d$

$$(\vec{x}+\mathbb{Z}^d) + (\vec{y}+\mathbb{Z}^d) = (\vec{x}+\vec{y})+\mathbb{Z}^d$$

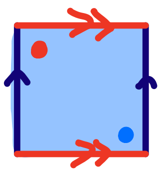
We endow $\mathbb{T}^d$ with the metric

$$d(\vec{x}+\mathbb{Z}^d, \vec{y}+\mathbb{Z}^d) = \min\{\|\vec{\xi}-\vec{\eta}\| : \vec{\xi}\in\vec{x}+\mathbb{Z}^d, \vec{\eta}\in\vec{y}+\mathbb{Z}^d\}$$

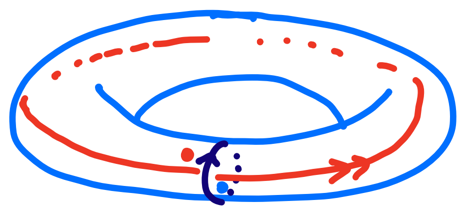
Two-Dimensional Torus: $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$



$\mathbb{R}^2/\mathbb{Z}^2$



$[0,1]^2/$identification

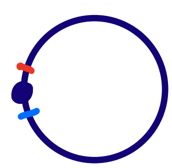


usual picture (homeomorphic, but not isometric because the curvature is misrepresented)

One Dimensional Torus: $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z} = S^1$

$\mathbb{R}/\mathbb{Z}$

$[0,1]/_{0\sim 1}$



S^1

Translation on $\mathbb{T}^1$: $R_\alpha(x+\mathbb{Z}) = (x+\alpha)+\mathbb{Z}$

(traditional notation: $R_\alpha(x) = x+\alpha \pmod 1$)

Thm (Dirichlet):

(1) If α is rational, then every orbit of R_α is periodic

(2) If α is irrational, then every orbit of R_α is dense (and R_α is minimal).

Proof.

(1) Suppose $\alpha = \frac{p}{q}$, $p,q \in \mathbb{Z}$. Then $R_\alpha^q(x) = x+p \pmod 1 = x$

(2) Suppose α is irrational. Fix $N \geq 1$ and divide $[0,1)$ into N equal intervals:

$$[0, \tfrac{1}{N}), \; [\tfrac{1}{N}, \tfrac{2}{N}), \; \dots, \; [\tfrac{N-1}{N}, 1).$$

Look at $\alpha, 2\alpha, \dots, (N+1)\alpha$.

By Dirichlet's Principle, $\exists\; 1 \leq i < j \leq N+1$ s.t. $i\alpha, j\alpha$ are in the same subinterval. So

$$0 < (j-i)\alpha < 1/N.$$

because $\alpha \notin \mathbb{Q}$ because $i\alpha, j\alpha$ are in the same interval

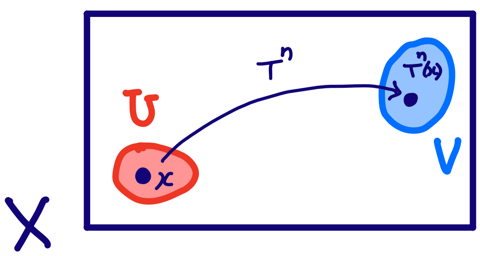
Thus $\{k\alpha \pmod 1 : k \geq 0\}$ is $\frac{1}{N}$-dense in $[0,1)$.

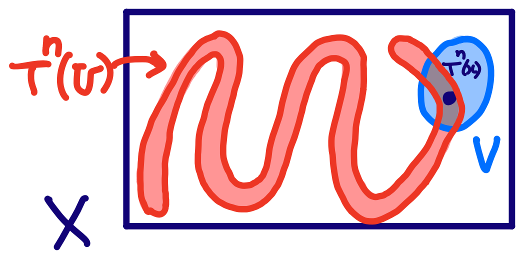
Since this is true for all $N \geq 1$, $\{k\alpha : k \geq 0\}$ is dense in $[0,1)$. So $\{x + k\alpha \pmod 1 : k \geq 0\}$ is dense too. $\square$

Topological Transitivity

Def$^{\underline{n}}$. A continuous map on a metric space is called **topologically transitive** if for all open sets $U, V \neq \emptyset$ there exists $n > 0$ s.t. $U \cap T^{-n} V \neq \emptyset$.

Recall: $T^{-n} V \equiv \{x : T(x) \in V\}$. Thus the condition $U \cap T^{-n} V \neq \emptyset$ says: "$\exists x \in U$ s.t. $T^n(x) \in V$"





Theorem. Suppose T is a continuous map on a complete separable metric space X (e.g. a compact metric space). If T is topologically transitive, then $\exists x_0 \in X$ s.t. $\overline{\{T^n(x_0) : n \geq 1\}} = X$.

["Complete": Every Cauchy sequence converges
"Separable": Contains a countable dense set.]

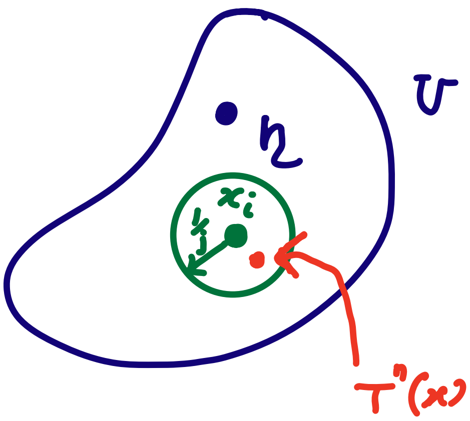
Exercise: Prove ($\Leftarrow$) under the additional assumption that X has no isolated points: x s.t. $\exists \epsilon > 0$ s.t. $d(y,x) < \epsilon \Rightarrow y = x$.

Proof of Thm: We use Baire's Category's Theorem (see below).

Since X is separable, there is a countable dense set of points $\overline{\{\xi_1, \xi_2, \xi_3, \ldots\}} = X$.

The forward orbit of x is dense in X iff

$(*)$ $\forall\, i,j\ \exists\, n>0$ s.t. $d(T^n(x), \xi_i) < 1/j$.



(because in this case $T^n(x)$ visits any neigh. of any point η)

Notice that $(*)$ holds iff $x \in \bigcap_{i,j=1}^{\infty} \left(\bigcup_{n=1}^{\infty} T^{-n} B(\xi_i, \tfrac{1}{j}) \right)$.

Thus it is sufficient to show that

$$(*) \qquad \bigcap_{i,j=1}^{\infty} \left(\bigcup_{n=1}^{\infty} T^{-n} B(\xi_i, \tfrac{1}{j}) \right) \neq \emptyset .$$

Notice that each $U_{ij} := \bigcup_{n=1}^{\infty} T^{-n} B(\xi_i, \frac{1}{j})$ is

(1) <u>open</u> (because T is continuous)

(2) and <u>dense</u> (because by top. transitivity it intersects any non-empty set)

By Baire's Category Theorem (see below), $(*)$ holds. $\square$

Baire's Category Theorem: Let (X,d) be a complete and separable metric space. Then any intersection of countably many open dense sets is dense (hence non-empty).

<u>Proof</u>. Let $U_1, U_2, U_3, \dots$ be a countable collection of open dense sets. Fix some arbitrary open ball B.

- U_1 is dense $\Rightarrow U_1 \cap B \neq \emptyset \Rightarrow \exists$ ball $\overline{B(x_1,r_1)} \subseteq U_1 \cap B$.
- U_2 is dense $\Rightarrow \exists$ ball $\overline{B(x_2,r_2)} \subseteq U_2 \cap B(x_1,r_1)$, $r_2 < \frac{1}{2} r_1$
- U_3 is dense $\Rightarrow \exists$ ball $\overline{B(x_3,r_3)} \subseteq U_3 \cap B(x_2,r_2)$, $r_3 < \frac{1}{2} r_2$

etc. By construction, $r_k \to 0$.

By construction, $\overline{B(x_1,r_1)} \supseteq \overline{B(x_2,r_2)} \supseteq \dots$

By completeness, $\bigcap_{i=1}^{\infty} \overline{B(x_i,r_i)} =$ single point x

[at <u>most</u> one point because $r_i \to 0$; at <u>least</u> one point because x_i is a Cauchy sequence whose limit $\in \bigcap \overline{B(x_i,r_i)}$]

Clearly, $x \in U_i \cap B$ for all $i \geq 1$.

So $x \in \bigcap_{i=1}^{\infty} U_i \cap B$.

In summary, $\bigcap_{i=1}^{\infty} U_i$ intersects any ball B. $\square$

Exercise: Suppose T is a homeomorphism of a complete and separable metric space, and for every $U, V \neq \emptyset$ open, $\exists n \in \mathbb{Z}$ s.t. $U \cap T^n(V) \neq \emptyset$. Then $\exists x_0 \in X$ s.t. $\{T^n(x_0) : n \in \mathbb{Z}\}$ is dense.

Example: Translations on $\mathbb{T}^d$

Let $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_d)$, $R_{\underline{\alpha}}(\underline{x} + \mathbb{Z}^d) = \underline{x} + \underline{\alpha} + \mathbb{Z}^d$

Kronecker's Theorem: The following are equivalent

(1) $1, \alpha_1, \alpha_2, \dots, \alpha_d$ are linearly independent over $\mathbb{Z}$
(i.e. if $\sum m_i \alpha_i = m_0$ with $m_0, m_1, \dots, m_d \in \mathbb{Z}$, then $m_i = 0$).

(2) $R_{\underline{\alpha}}$ is topologically transitive.

(3) $R_{\underline{\alpha}}$ is minimal.

Proof:

(1) $\Rightarrow$ (2). First we claim that (1) implies the existence of $\underline{x}_0$ s.t. $\{R_{\underline{\alpha}}^n(\underline{x}_0) : n \in \mathbb{Z}\}$ is dense (this is weaker than top transitivity which claims the existence of $\underline{y}_0$ s.t. $\{R_{\underline{\alpha}}^n(\underline{y}_0) : n \in \mathbb{N}\}$ is dense).

Assume by way of contradiction that there's no $\underline{x}_0$ s.t. $\{R^n_{\underline{\alpha}}(\underline{x}_0) : n \in \mathbb{Z}\}$ is dense. By the exercise, $\exists\, U, V$ open and non-empty s.t. $U \cap R^n_\alpha V = \emptyset \ \forall n \in \mathbb{Z}$.

Equivalently, $U \cap \bigcup_{n\in\mathbb{Z}} R^n_\alpha V = \phi$.

Let $\chi : \mathbb{T}^d \to \{0,1\}$, $\chi(\underline{x}) = \begin{cases} 1 & \underline{x} \in \bigcup_{n\in\mathbb{Z}} R^n_{\underline{\alpha}}(V) \\ 0 & \text{otherwise} \end{cases}$.

Then:

(1) $\chi \in L^2(\mathbb{T}^d)$

(2) $\chi \neq \text{const}$ a.e. (because $\chi = 1$ on V, $\chi = 0$ on U)

(3) $\chi \circ R_{\underline{\alpha}} = \chi$

But this is impossible: Consider the Fourier expansion

$$\chi(\underline{x}) = \sum_{\underline{m}\in\mathbb{Z}^d} \hat{\chi}(\underline{m})\, e^{2\pi i \langle \underline{m}, \underline{x} \rangle}$$

$$\|$$

$$\chi(\underline{x}+\underline{\alpha}) = \sum_{\underline{m}\in\mathbb{Z}^d} e^{2\pi i \langle \underline{m}, \underline{\alpha} \rangle}\, \hat{\chi}(\underline{m})\, e^{2\pi i \langle \underline{m}, \underline{x} \rangle}$$

Equating coefficients we obtain

$$\boxed{\hat{\chi}(\underline{m}) = e^{2\pi i \langle \underline{m}, \underline{\alpha} \rangle}\, \hat{\chi}(\underline{m})}$$

So either

(a) $\underline{m} = 0$, or

(b) $\underline{m} \neq 0$, whence by (1) $\langle \underline{m}, \underline{\alpha} \rangle \neq 0$, whence $\hat{\chi}(\underline{m}) = 0$.

Thus $\hat{\chi}(\underline{x}) = \hat{\chi}(\underline{0}) = \text{const.}$, a contradiction.

The contradiction shows that $\exists \underline{x}_0$ s.t. $\{R_{\underline{\alpha}}^n(\underline{x}_0) : n \in \mathbb{Z}\}$ is dense.
Since $\{R_{\underline{\alpha}}^n(\underline{x}_0) : n \in \mathbb{Z}\} = \{\underline{x}_0 + n\underline{\alpha} : n \in \mathbb{Z}\}$, $\{n\underline{\alpha} : n \in \mathbb{Z}\}$ is dense.

Next, we claim that $R_{\underline{\alpha}}$ is topologically transitive.

Take U, V open and non-empty. Choose $m, n \in \mathbb{Z}$ s.t. $R_{\underline{\alpha}}^m(\underline{0}) \in U$, $R_{\underline{\alpha}}^n(\underline{0}) \in V$.
If $n > m$, $U \cap R_{\underline{\alpha}}^{-k} V \neq \phi$ for $k := n - m$, $k > 0$, and we're done.

If $n \leq m$, we need to do something. By the denseness of $\{R_{\underline{\alpha}}^n(\underline{0}) : n \in \mathbb{Z}\}$, $\exists k_i \in \mathbb{Z}$ s.t. $|k_i| \to \infty$, $R_{\underline{\alpha}}^{k_i}(\underline{0}) \to \underline{0}$.
Without loss of generality $k_i \to \infty$ or $k_i \to -\infty$.

- If $k_i \to \infty$, then $R_{\underline{\alpha}}^{n+k_i}(\underline{0}) \longrightarrow R_{\underline{\alpha}}^{n}(\underline{0}) \in V$ and for some i, $\bar{n} := n+k_i > m$, $R_{\underline{\alpha}}^{\bar{n}}(\underline{0}) \in V$, $R_{\underline{\alpha}}^{m}(\underline{0}) \in U$
- If $k_i \to -\infty$, then $R_{\underline{\alpha}}^{m+k_i}(\underline{0}) \to R_{\underline{\alpha}}^{m}(\underline{0}) \in U$ and for some i, $\bar{m} = m+k_i < n$, $R_{\underline{\alpha}}^{\bar{m}}(\underline{0}) \in U$, $R_{\underline{\alpha}}^{n}(\underline{0}) \in V$

So we found $N > M$ s.t. $R_{\underline{\alpha}}^{N}(\underline{0}) \in V$, $R_{\underline{\alpha}}^{M}(\underline{0}) \in U$ as needed.

So $(1) \Rightarrow (2)$.

<u>Proof that $(2) \Rightarrow (3)$</u>: Suppose $R_{\underline{\alpha}}$ is top. transitive. We show it's minimal.

Top. transitivity guarantees the existence of $\underline{x}_0$ s.t. $\{R_{\underline{\alpha}}^{n}(\underline{x}_0) : n \in \mathbb{N}\}$ is dense.

$\Rightarrow \{\underline{x}_0 + n\underline{\alpha} : n \in \mathbb{N}\}$ is dense

$\Rightarrow \{n\underline{\alpha} : n \in \mathbb{N}\}$ is dense

$\Rightarrow \{\underline{x} + n\underline{\alpha} : n \in \mathbb{N}\}$ is dense for all $\underline{x} \in \mathbb{T}^d$

$\Rightarrow R_{\underline{\alpha}}$ is minimal.

(3) $\Rightarrow$ (1) Suppose $R_{\underline{\alpha}}$ is minimal. We show that $1, \alpha_1, \dots, \alpha_d$ are independent over $\mathbb{Z}$. Assume $m_1, \dots, m_d \in \mathbb{Z}$ and $\sum m_i \alpha_i \in \mathbb{Z}$, we have to show that $m_1 = \dots = m_d = 0$.

Define $F : \mathbb{T}^d \to \mathbb{R}$

$$F(\underline{x}) = \exp\left(2\pi i \sum_{i=1}^{d} m_i x_i\right)$$

This is a continuous function, and $F \circ R_{\underline{\alpha}} = F$, because

$$F(\underline{x}+\underline{\alpha}) = \exp\left(2\pi i \sum_{i=1}^{d} m_i x_i\right) \exp\left(2\pi i \sum_{i=1}^{d} m_i \alpha_i\right)$$

$$= F(\underline{x}) \cdot 1 = F(\underline{x}).$$

By invariance, F is constant on $\{n\underline{\alpha} : n \in \mathbb{N}\}$. This set is dense, and F is continuous. It follows that $F = \text{const}$.

But if F is constant, then

$$0 = \frac{\partial F}{\partial x_i}(\underline{0}) = 2\pi i \, m_i$$

So $m_1 = \dots = m_d = 0$. $\square$

Exercises

(1) Let (X,d) be a metric space. An <u>isolated point</u> is a point $x \in X$ s.t. for some $r>0$,

$$B(x,r) := \{y \in X : d(x,y) < r\}$$

equals $\{x\}$. Show: If T has a dense orbit and X has no isolated points, then $\exists x \in X$ with ω-limit set

$$\omega(x) = X.$$

(2) Give an example of a continuous map on a metric space with isolated points for which there is a dense orbit, but $\omega(x) = \emptyset$ for all $x \in X$.

(3) Prove direction $(\Leftarrow)$ in the topological transitivity thm: If (X,d) has no isolated points and $\exists x$ s.t. $\{T^n(x) : n \geq 1\}$ is dense in X, then T is topologically transitive.

(Be carefull: the n in $U \cap T^{-n} V \neq \emptyset$ should be positive).

(4) (a) Suppose α is irrational. Show that $\exists s > 0$ s.t. $(1, s, \alpha s)$ are independent over $\mathbb{Z}$.

(b) Prove that if $\vec{v} \in \mathbb{R}^2$ has irrational slope, then the half line $\{p + t\vec{v} : t \geq 0\}$ is dense on the torus $\mathbb{T}^2$ for all $p \in \mathbb{T}^2$.

(c) Suppose $\vec{v} \in \mathbb{R}^2$ has rational slope. Prove that the half line $\{p + t\vec{v} : t \geq 0\}$ forms a loop. Describe its length.