Lecture 7: Ergodicity

Recall from the previous lecture:

- A function f is T-invariant, if $f \circ T = f$
- A measure μ is T-invariant, if $\mu \circ T^{-1} = \mu$
- A measure μ is ergodic, if any T-inv function is equal almost everywhere (a.e.) to a constant

Ergodic Thm: If μ is T-invariant and ergodic, then for every bounded measurable function f,

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \longrightarrow \int_X f\,d\mu \quad \text{a.e.}$$

What about the non-ergodic case?

Ergodic Decomposition: Suppose T is a continuous map on a compact metric space* X. Any T-invariant Borel prob measure μ can be put in the form

$$(*) \quad \mu = \int_X \nu_\xi \, d\mu(\xi)$$

, where ν_ξ are T-inv ergodic Borel measures.

This means that for every bounded Borel $f: X \to \mathbb{R}$,

$\xi \mapsto \int_X f\,d\nu_\xi$ is Borel measurable, and $\int_X f\,d\mu = \int_X \left[\int_X f\,d\nu_\xi\right] d\mu$

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* There are more general versions of this theorem. The proof can be found in my lecture notes on Ergodic theory. I skip it because it requires more measure theory than I'm willing to assume.

The measures μ_ξ are called "ergodic components".

Corollaries:

(1) If $\exists$ inv prob measures, $\exists$ ergodic inv prob measures. So:

"Ergodic" Krylov-Bogolyubov Thm: Any continuous map on a compact metric space has at least one ergodic invariant prob. measure.

(2) If all ergodic inv prob. measures are equal, then all non-necessarily ergodic inv measures are equal (because (*) has just one value). Thus:

$$\begin{pmatrix}\text{unique}\\ \text{ergodicity}\end{pmatrix} \equiv \begin{pmatrix}\text{unique}\\ \text{invariant}\\ \text{measure}\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\exists \text{ unique } \underline{\text{ergodic}}\\ \text{invariant measure}\end{pmatrix}.$$

(3) Suppose μ is a non-ergodic but T-invariant measure on X. Then $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k x)$ exists μ-a.e., (but the limit is not necessarily $\int f d\mu$).

Exercise: Prove this, by showing that

$$\{x \in X : \text{ the limit exists}\}$$

has full measure for each ergodic component.

An Important Limitation of the Ergodic Thm: Formally what this theorem says is that the set

$$\Omega_f(\mu) = \left\{ x \in X : \frac{1}{N} \sum_{k=0}^{N-1} f(T^k x) \longrightarrow \int_X f \, d\mu \right\}$$

has full measure. But

- Ω_f may depend on f
- The theorem does not tell us which x belong to Ω_f and which do not.
- If there are two ergodic inv measures μ_1, μ_2 then $\Omega_f(\mu_1) \cap \Omega_f(\mu_2) = \emptyset$ whenever $\int f d\mu_1 \neq \int f d\mu_2$, so $\Omega_f(\mu_i) \neq X$.

The Problem: In non-uniquely ergodic cases Ω_f depends on f.

To deal with this we introduce the following definition:

<u>Def$^{\underline{n}}$</u>. Suppose $T: X \to X$ is a continuous map on a compact metric space, with an invariant probability measure μ.

A point $x_0 \in X$ is called **μ-generic**, if for every <u>continuous</u> function $f \in C(X)$

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n(x_0)) \longrightarrow \int f \, d\mu .$$

<u>Thm</u>. Suppose μ is ergodic, then $\exists$ μ-generic points. Indeed

$$\mu \{ x \in X : x \text{ is } \mu\text{-generic} \} = 1 .$$

<u>Proof</u>. Since X is compact and metric, $C(X)$ is separable: $\exists$ a <u>countable</u> collection of $f_n \in C(X)$ s.t. $\forall f \in C(X)$ $\forall \varepsilon$ $\exists f_n$ s.t. $\| f - f_n \|_\infty := \sup_X |f - f_n| < \varepsilon$.

By the ergodic thm, for each n,

$$\Omega_n := \left\{ x \in X : \frac{1}{N} \sum_{k=1}^{N} f_n(T^k(x)) \xrightarrow[N\to\infty]{} \int f_n \, d\mu \right\}$$

has full measure. Therefore

$$\Omega := \bigcap_{n=1}^{\infty} \Omega_n \quad \text{has full measure}$$

$$\left(\text{proof: } \mu(\Omega^c) = \mu\left(\bigcup_{n=1}^{\infty} \Omega_n^c \right) = \mu\left(\biguplus_{n=1}^{\infty} \left(\Omega_n \setminus \bigcup_{j=0}^{n-1} \Omega_j \right) \right) \right.$$

$$\left. = \sum_{n=1}^{\infty} \mu\left(\Omega_n \setminus \bigcup_{j=0}^{n-1} \Omega_j \right) \le \sum_{n=1}^{\infty} \mu(\Omega_n) = \sum_{n=1}^{\infty} 0 = 0. \right)$$

Now suppose $x_0 \in \Omega$ and $f \in C(X)$.

For every ϵ, $\exists f_n$ s.t. $\|f - f_n\| < \epsilon$.

So $f_n - \epsilon \le f \le f_n + \epsilon$, and so

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T^k x_0) \le \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f_n(T^k x_0) + \epsilon$$

$$\le \int f_n \, d\mu + \epsilon \le \int (f + \epsilon) \, d\mu + \epsilon \le \int f d\mu + 2\epsilon.$$

Similarly, $\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T^k(x_0)) \ge \int f d\mu - 2\epsilon$.

Since ϵ is arbitrary, $\limsup = \liminf = \int f d\mu$.

As f was arbitrary, x_0 is generic. $\square$

<u>Exercise</u>: If $\frac{1}{N} \sum_{k=0}^{N-1} f(T^k x) \to \int_X f \, d\mu \quad \forall f \in \begin{pmatrix} \text{dense subset} \\ \text{of } C(X) \end{pmatrix}$

Then x is μ-generic.

<u>Exercise In Measure Theory</u>: Suppose μ is a Borel probability measure on a compact metric space (X,d), and let

$$B(x,r) = \{ y \in X : d(x,y) < r \}.$$

$$\partial B(x,r) = \{ y \in X : d(x,y) = r \}$$

(a) There are <u>at most</u> countably many $r>0$ s.t. $\mu(\partial B(x,r)) \neq 0$.

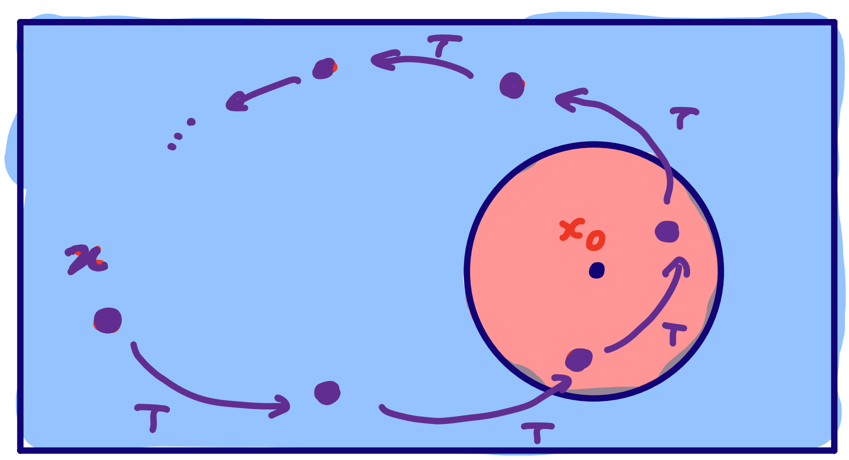
(b) If $\mu[\partial B(x,r)] = 0$, then $\forall \varepsilon > 0$ $\exists f, g \in C(X)$ s.t. $g \le 1_{B(x,r)} \le f$ and $\int |f-g| \, d\mu < \varepsilon$

(c) f, g must also satisfy $|\int f - \mu(B(x,r))|, |\int g - \mu(B(x,r))| < \varepsilon$.

<u>Thm</u>. Suppose x is a μ-generic point for a continuous, ergodic measure preserving T on a compact metric space X. For each $x_0 \in X$, for all but countably many $r > 0$,

$$\underbrace{\frac{1}{N} \# \{ 1 \le n \le N : T^n(x) \in B(x_0, r) \}}_{\substack{\text{frequency of visits} \\ \text{to } B(x,r)}} \xrightarrow[N \to \infty]{} \mu(B(x_0, r))$$

<u>Exercise</u>: Prove this.



We'll explore applications of this to number theory.

Preparations: Generic Points for Skew Products

Def$^{\underline{n}}$. Suppose $T: X \to X$ is a continuous map on a compact metric space, and $\varphi: X \to \mathbb{T}$ is continuous. The **skew product** (with "base" T and "cocycle" φ) is the continuous map $T_\varphi: X \times \mathbb{T} \to X \times \mathbb{T}$

$$T_\varphi(x,t) = (T(x), t + \varphi(x))$$

__Exercise__: Prove by induction that

$$T_\varphi^n(x,t) = \left(T^n(x), t + \sum_{k=0}^{n-1} \varphi(T^k x)\right)$$

__Exercise__: Suppose μ is a Borel measure on X and λ is Lebesgue's measure on $\mathbb{T}$.

(1) There is a unique Borel measure $m := \mu \times \lambda$ on $X \times \mathbb{T}$ s.t.

$$\int_{X \times \mathbb{T}} F \, d(\mu \times \lambda) = \int_X \left(\int_{\mathbb{T}} F(x,t) \, dt \right) d\mu(x) \qquad (F \in C(X \times \mathbb{T}))$$

(2) If μ is T-invariant, then m is T_φ-invariant.

__Furstenberg's Thm__. Suppose $T: X \to X$ is a uniquely ergodic map on a compact metric space with invariant prob. measure μ. If $\mu \times \lambda$ is T_φ-ergodic, then T_φ is __uniquely__ ergodic.

Proof: Let $m := \mu \times \lambda$. The crucial observation is:

Claim: For all $t_1, t_2 \in \mathbb{T}$, (x, t_1) is m-generic iff (x, t_2) is m-generic.

Proof. Let $Q_a : X \times \mathbb{T} \to X \times \mathbb{T}$ be the map $Q_a(x,t) = (x, t+a)$. Note that Q_a is continuous, and $Q_a \circ T_\varphi = T_\varphi \circ Q_a$.

Suppose (x, t_1) is m-generic. Then $\forall F \in C(X \times \mathbb{T})$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (F \circ T_\varphi^n)(x, t_2)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (F \circ T_\varphi^n \circ Q_{t_2 - t_1})(x, t_1)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} ((F \circ Q_{t_2-t_1}) \circ T_\varphi^n)(x, t_1)$$

$$= \int_{X\times\mathbb{T}} (F \circ Q_{t_2-t_1})\, dm,$$ by m-genericity with the test function $F \circ Q_{t_2 - t_1}$.

$$= \int_X \left(\int_{\mathbb{T}} F(x, t + t_2 - t_1)\, dt \right) d\mu(x) = \int_X \int_{\mathbb{T}} F(x,t)\, dt\, d\mu(x)$$

$$= \int_{X\times\mathbb{T}} F\, dm.$$ So (x, t_2) is m-generic.

Thus (x, t_1)-genericity $\Longrightarrow$ (x, t_2)-genericity. By symmetry the two are equivalent.

By the claim, the set of m-generic points must have the form

$$\Omega_m := \{(x,t) : \text{m-generic}\} = E_\mu \times \mathbb{T}, \text{ where}$$

$E_\mu \subseteq X$ is given by $E_\mu := \{x : (x,0) \text{ is m-generic}\}$.

[Exercise: Prove that E_μ is Borel!] Note that

$$\mu(E_\mu) = \int_X 1_{E_\mu}\, d\mu = \int_X \int_{\mathbb{T}} 1_E(x,t)\, dt\, d\mu(x) = m(E_\mu \times \mathbb{T}) = m(\Omega_m) = 1.$$

So

$$\boxed{\mu(E_\mu) = 1}$$

Now, suppose $\tilde{m}$ is some other ergodic T_φ-inv prob. measure. Let $\tilde{\mu} := \tilde{m} \circ \pi^{-1}$, where $\pi : X \times \mathbb{T} \to X$ is the projection $\pi(x,t) = x$.

(If $E \subseteq X$, then $\pi^{-1}(E) \subseteq X \times \mathbb{T}$ so $\tilde{m}(\pi^{-1}E)$ is defined).

Claim: $\tilde{\mu} = \mu$

Proof: Since T is uniquely ergodic, it's sufficient to show that $\tilde{\mu} \circ T^{-1} = \tilde{\mu}$. Fix some $f : X \to \mathbb{R}$ continuous, then

$$\int_X f \circ T\, d\tilde{\mu} = \int_X (f \circ T)\, d\tilde{m} \circ \pi^{-1} = \int_X (f \circ T) \circ \pi\, d\tilde{m}$$

(see lecture 5)

$$= \int_X (f\circ\pi)\circ T_\varphi \, d\tilde{m} \qquad (\text{check that } \pi\circ T_\varphi = T\circ\pi)$$

$$= \int_X f\circ\pi \; d\tilde{m}\circ T_\varphi^{-1} \underset{\tilde{m}\circ T_\varphi^{-1} = \tilde{m}}{=} \int f\circ\pi \, d\tilde{m} = \int f \, d\tilde{\mu}.$$

So $\tilde{\mu}\circ T^{-1} = \tilde{\mu}$. By unique ergodicity $\tilde{\mu} = \mu$.

We now compare the $\tilde{m}$-measures of

$$\Omega_{\tilde{m}} := \{ (x,t) : \tilde{m}\text{-generic} \}$$

$$\Omega_m := \{ (x,t) : m\text{-generic} \} = E_\mu \times \mathbb{T}.$$

- $\tilde{m}(\Omega_{\tilde{m}}) = 1$, because $\tilde{m}$-a.e. (x,t) is $\tilde{m}$-generic.
- $\tilde{m}(\Omega_m) = \tilde{m}(E_\mu \times \mathbb{T}) = \tilde{m}(\pi^{-1}(E_\mu)) = (\tilde{m}\circ\pi^{-1})(E_\mu)$

$$= \tilde{\mu}(E_\mu) \underset{\text{claim}}{=} \mu(E_\mu) = 1$$

$\Rightarrow \Omega_{\tilde{m}} \cap \Omega_m \neq \emptyset$ (otherwise $\tilde{m}(X\times\mathbb{T}) \geq 2$)

$\Rightarrow \exists (x,t)$ s.t. $\forall F \in C(X\times\mathbb{T})$

$$\int_{X\times\mathbb{T}} F \, d\tilde{m} \underset{(x,t)\in\Omega_{\tilde{m}}}{=} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (F\circ T_\varphi^n)(x,t) \underset{(x,t)\in\Omega_m}{=} \int_{X\times\mathbb{T}} F \, dm$$

$\Rightarrow \tilde{m} = m$. In summary, all <u>ergodic</u> T_φ-inv prob. measures are equal to m. As explained before, this implies unique ergodicity. $\square$