Lecture 8: Uniform Distribution

Def$^{\underline{n}}$. A sequence of $x_n \in [0,1]$ is called uniformly distributed (or equidistributed) in $[0,1]$, if for any subinterval $(\alpha,\beta) \subseteq [0,1]$,

$$\frac{1}{N} \# \{ 1 \le n \le N : \; x_n \in (\alpha,\beta) \} \xrightarrow[N\to\infty]{} |\alpha-\beta| .$$

Our aim is to prove the following:

Weyl's Equidistribution Thm: Suppose α is irrational and $k \in \mathbb{N}$. Then $\{n^k\alpha\}$ = fractional part of $n^k\alpha$ is uniformly distributed in $[0,1]$.

Proof for $k=1$: This is the easiest case.
Consider the map $R_\alpha : \mathbb{T} \to \mathbb{T}$, $R_\alpha(x) = x+\alpha \pmod 1$.
This map preserves Lebesgue's measure λ.

Claim: R_α is uniquely ergodic.

Proof. It's convenient to introduce the following notation for $u,v : \mathbb{T} \to \mathbb{R}$:

$$\int_{\mathbb{T}} (u+iv)\, d\mu := \int u \, d\mu + i \int v \, d\mu .$$

For example, $\displaystyle\int_{\mathbb{T}} e^{2\pi i n t} d\lambda$

$$= \int_0^1 \cos(2\pi n t)\,dt + i \int_0^1 \sin(2\pi n t)\,dt = \begin{cases} 1 & n=0 \\ 0 & n \in \mathbb{Z}\setminus\{0\} \end{cases} .$$

The key observation here is that we can calculate the ergodic sum $\sum_{k=0}^{N-1} (\chi_n \circ R_\alpha^k)(x)$ for all $\chi_n(t) = e^{2\pi i n t}$:

<u>Case 1 ($n=0$)</u>: $\chi_n \equiv 1$, so $\frac{1}{N}\sum_{k=0}^{N-1} \chi_n \circ R_\alpha^k \equiv 1 = \int_{\mathbb{T}} \chi_0 \, d\lambda$

<u>Case 2 ($n \neq 0$)</u>:

$$\frac{1}{N}\sum_{k=0}^{N-1} (\chi_n \circ R_\alpha^k)(t) = \frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i n (t+k\alpha)}$$

$$= \frac{1}{N} e^{2\pi i n t} \sum_{k=0}^{N-1} \left(e^{2\pi i n \alpha}\right)^k = \frac{1}{N} e^{2\pi i n t} \left(\frac{e^{2\pi i n N\alpha} - 1}{e^{2\pi i n \alpha} - 1} \right)$$

Thus, $$\left| \frac{1}{N} \sum_{k=0}^{N-1} \chi_n \circ R_\alpha^k \right| \leq \frac{1}{N} \frac{|e^{2\pi i n N\alpha}| + 1}{|e^{2\pi i n \alpha} - 1|}$$

$$\leq \frac{1}{N} \cdot \frac{2}{|e^{2\pi i n \alpha} - 1|} \xrightarrow[N\to\infty]{} 0 = \int_{\mathbb{T}} \chi_n \, d\lambda$$

In summary, for all $n \in \mathbb{Z}$,

$$\frac{1}{N}\sum_{k=0}^{N-1} \chi_n \circ R_\alpha^k \xrightarrow[N\to\infty]{} \int_{\mathbb{T}} \chi_n \, d\lambda \quad \underline{\text{uniformly}} \text{ on } \mathbb{T}.$$

By linearity, for every $p(t) \in \mathrm{Span}\{\chi_n : n \in \mathbb{Z}\} =: \mathrm{Trig}$ ("trigonometric polynomial"),

$$\frac{1}{N}\sum_{k=0}^{N-1} p \circ R_\alpha^k \xrightarrow[N\to\infty]{} \int_{\mathbb{T}} p \, d\lambda \quad \underline{\text{uniformly}} \text{ on } \mathbb{T}.$$

By the <u>Stone-Weierstrass theorem</u>, the trigonometric polynomials are dense in $C(\mathbb{T})$:

$$\forall \varepsilon>0\ \forall f\in C(X)\ \exists p\in \mathrm{Trig}\ \left(\forall x\in \mathbb{T}\ \ |f(x)-p(x)|<\varepsilon\right).$$

$\underset{\text{exercise}}{\Longrightarrow}$ $\forall f\in C(X)$, $\displaystyle\frac{1}{N}\sum_{k=0}^{N-1} f\circ R_\alpha^k \xrightarrow[N\to\infty]{} \int_{\mathbb{T}} f\,d\lambda$ uniformly on $\mathbb{T}$.

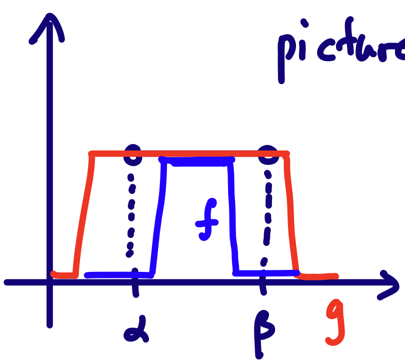
In other words, any $x\in\mathbb{T}$ is λ-generic.
Necessarily R_α is uniquely ergodic (if there were some other ergodic inv measure ν, its generic points wouldn't be λ-generic: Consider what happens for f s.t. $\int f\,d\lambda \neq \int f\,d\nu$)

In particular, $x=0$ is λ-generic, whence, by an exercise

$$\frac{1}{N}\#\{1\le n\le N:\ R_\alpha^n(0)\in B(x_0,r)\} \xrightarrow[N\to\infty]{} B(x_0,r)$$

for any open ball $B(x_0,r)$ s.t. $\lambda(\partial B(x_0,r))=0$.
All balls are like that, e.g. $(a,b) := B\left(\frac{a+b}{2},\frac{b-a}{2}\right)$.

<u>Direct Proof</u>: Sandwich $f\le 1_{(a,b)}\le g$ as in the picture s.t. $f,g\in C(\mathbb{T})$, and $\int f,\int g$ ε-close to $a-b$. Then



$$\frac{1}{N}\sum_{n=0}^{N-1} f(R_\alpha^n(0)) \le \frac{1}{N}\#\{1\le n\le N:\ R_\alpha^n(0)\in(a,b)\} \le \frac{1}{N}\sum_{n=0}^{N-1} g(R_\alpha^n(0))$$

Observe: $\lim\limits_{N\to\infty}$ LHS, $\lim\limits_{N\to\infty}$ RHS are ε-close to $|a-b|$.

$\Longrightarrow$ $\limsup, \liminf \frac{1}{N}\#\{1\le n\le N:\ R_\alpha^n(0)\in(a,b)\}$ are ε-close to $|a-b|$.
Since ε is arbitrary, $\limsup=\liminf=|a-b|$. $\square$

Proof for $k>1$: The idea is to "manufacture" $\{n^k\alpha\}$ from sum uniquely ergodic map. We'll do this in detail next time. Now we just demonstrate the key idea, looking at $k=2$.

Define $T_\beta(x,y) = (x+\beta, y+x)$ on $\mathbb{T}\times\mathbb{T}$.

$$T_\beta^2(x,y) = (x+2\beta,\ y+x+x+\beta) = (x+2\beta,\ y+2x+\beta)$$

$$T_\beta^3(x,y) = T_\beta(x+2\beta,\ y+2x+\beta) = (x+3\beta,\ y+3x+\beta+2\beta)$$

$$T_\beta^3(x,y) = T_\beta(x+3\beta,\ y+3x+\beta+2\beta)$$

$$= (x+4\beta,\ y+4x+\beta+2\beta+3\beta)$$

. . . . .

$$T_\beta^n(x,y) = (x+4\beta,\ y+nx+\underbrace{\beta+2\beta+\dots+(n-1)\beta}_{=\binom{n}{2}\beta = (n^2-n)\cdot\beta/2})$$

Choosing $\boxed{\beta = 2\alpha,\ x=\alpha,\ y=0}$,

we obtain

$$T_\beta^n(\alpha,0) = \left((n+1)\alpha,\ n\alpha + (n^2-n)\cdot\frac{2\alpha}{2}\right) = \left((n+1)\alpha,\ n^2\alpha\right).$$

If we can show unique ergodicity, we can deduce that $(\alpha,0)$ is generic, whence $\{n^2\alpha\}$ is uniformly distributed on $\mathbb{T}$.

Additional skew product extensions give higher powers.

Proof of Uniform Distribution of $\{n^k\alpha\}$ for $k\geq 1$

Step 1: A Map with an Orbit Which Generates $\{n^k\alpha\}$:

A Formula from Linear Algebra: Let

$$A = \begin{pmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 1 \end{pmatrix} \Big\updownarrow k+1$$

i = row (y-coord)
j = column (x-coord)

then

$$A^n = \begin{pmatrix} 1 & & & & & 0 \\ \binom{n}{1} & 1 & & & & \\ \binom{n}{2} & \binom{n}{1} & 1 & & & \\ \binom{n}{3} & \binom{n}{2} & \binom{n}{1} & 1 & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \binom{n}{k} & \cdots\cdots & & \binom{n}{2} & \binom{n}{1} & 1 \end{pmatrix} = \left(\binom{n}{i-j}\right)_{i,j=1,\dots,k+1}$$

with the convention $\binom{n}{k}=0$ for $k>n$.

Proof: For $n=1$, this is because $\binom{1}{0}=1$, $\binom{1}{1}=1$, $\binom{1}{k}=0$, $k\geq 2$.

Assume by induction the formula holds for n. Then

$$(A^{n+1})_{ij} = (A\cdot A^n)_{ij} = \sum_{l=1}^{k} A_{il}(A^n)_{lj} = A_{i,i-1}(A^n)_{i-1,j} + A_{i,i}(A^n)_{i,j}$$

$$= \binom{n}{i-j-1} + \binom{n}{i-j} = \begin{cases} \binom{n+1}{i-j} & i-j\leq n \\ \binom{n}{n} = \binom{n+1}{i-j} & i-j = n+1 \\ 0 & i-j\geq n+2 \end{cases}$$

$= \binom{n+1}{i-j}$ in all cases. $\square$

A Map with an Orbit Which Generates $\{n^k\alpha\}$

: Fix some β (to be determined later), and define $T := T_{\beta,k}$ by $T : \{\beta\}\times\mathbb{T}^k \to \{\beta\}\times\mathbb{T}^k$,

$$T(\beta; x_1, \dots, x_k) = (\beta; x_1+\beta, x_2+x_1, x_3+x_1, \dots, x_k+x_{k-1}) \bmod 1$$
$$= A\begin{pmatrix}\beta\\ x_1\\ \vdots\\ x_k\end{pmatrix} \bmod 1 \quad \text{for } A \text{ as before.}$$

Claim. $\exists \beta$ irrational and $\exists$ initial condition $(\beta; \xi_1, \dots, \xi_k)$ s.t.

$$T^n(\beta; \xi_1, \dots, \xi_k) = (\beta; *, \dots, *, \{n^k\alpha\})$$

Proof. Observe that $T^n(\beta; x_1, \dots, x_k) = A^n\begin{pmatrix}\beta\\ x_1\\ \vdots\\ x_k\end{pmatrix} \bmod 1.$

Since the last row of A^n is $\left(\binom{n}{k}, \dots, \binom{n}{1}, 1\right)$, the last coordinate of $T^n(\beta; x_1, \dots, x_k)$ equals

$$\binom{n}{k}\beta + \binom{n}{k-1}x_1 + \dots + \binom{n}{1}x_{k-1} + x_k .$$

As functions of n, $P_{k,j}(n) := \binom{n}{j} = \dfrac{n(n-1)\cdots(n-j+1)}{j!}$

are deg j polynomials in n. Since they have different degrees they are linearly independent. Dimension considerations give:

$$\mathrm{Span}\{1, P_{k,1}(t), \dots, P_{k,k}(t)\} = \mathrm{Span}\{1, t, \dots, t^k\}.$$

So $\exists \beta; \xi_1, \dots, \xi_k$ s.t.

$$\forall n \quad \binom{n}{k}\beta + \binom{n}{k-1}\xi_1 + \dots + \binom{n}{1}\xi_{k-1} + \xi_k = n^k\alpha .$$

Observe that α = leading coefficient $= \beta/k!$, so $\beta \notin \mathbb{Q}$. $\square$

Step 2: $T_{\beta,k}$ is uniquely ergodic for all $k \geq 1$, $\beta \notin \mathbb{Q}$.

Key Observation: T is a "tower" of skew products

$$T_{\beta,k+1}(\beta; x_1, \dots, x_k, x_{k+1}) =$$

$$= (\underbrace{\beta; x_1+\beta, x_2+x_1, \dots, x_k+x_{k-1}}_{T_{\beta,k}(\beta; x_1, \dots, x_k)}, \underbrace{x_{k+1}+x_k}_{x_{k+1}+\varphi_k(\beta; x_1, \dots, x_k)})$$

Therefore, it's natural to use induction on k, and Furstenberg's theorem on skew products.

$\underline{k=1}$: $T_{\beta,1}: \{\beta\} \times \mathbb{T} \to \{\beta\} \times \mathbb{T}$, $T_{\beta,1}(\beta; x_1) = (\beta, x_1+\beta)$.

This map is uniquely ergodic, because for every x_1 and every continuous function $f(\beta; t)$ on $\{\beta\} \times \mathbb{T}$,

$$\frac{1}{N} \sum_{n=0}^{N-1} (f \circ T_{\beta,1}^n)(\beta, x) = \frac{1}{N} \sum_{n=0}^{N-1} f(\beta; x+n\beta) \longrightarrow \int_0^1 f(\beta, t)\, dt, \text{ uniformly}$$

on $\beta \times \mathbb{T}$, by the unique ergodicity of the map $R_\beta: \mathbb{T} \to \mathbb{T}$, $R_\beta(t) = t+\beta \bmod 1$ (recall that $\beta \notin \mathbb{Q}$).

We also see that the unique invariant measure of $T_{\beta,1}$ is m_1 s.t.

$$\int_{\{\beta\} \times \mathbb{T}} f \, dm_1 = \int_0^1 f(\beta; t)\, dt \qquad (f \in C(\beta \times \mathbb{T})).$$

Induction Step. Assume by induction that $T_{\beta,k}$ is uniquely ergodic with the unique invariant measure m_k s.t.

$$\int_{\beta\times\mathbb{T}^k} f\,dm_k = \int_0^1 \cdots \int_0^1 f(\beta;t_1,\dots,t_k)\,dt_1 \cdots dt_k \quad (f\in C(\beta\times\mathbb{T}^k))$$

We'll show that $T_{\beta,k+1}$ is uniquely ergodic with invariant measure m_{k+1}. Furstenberg's thm says that it's enough to check the following conditions:

(a) $T_{\beta,k}$ is uniquely ergodic with inv measure m_k

(b) $T_{\beta,k+1}$ is ergodic with respect to $m_k\times\lambda \equiv m_{k+1}$.

Condition (a) is our induction hypothesis. It remains to check (b). To do this we must show that if $f(\beta;x_1,\dots,x_{k+1})$ is a bounded measurable <u>invariant</u> function (i.e. $f\circ T_{\beta,k+1} = f$ a.e.), then $f=\text{const. a.e.}$

Consider the function on $\mathbb{T}^{k+1}$

$$F(x_1,\dots,x_{k+1}) = f(\beta;x_1,\dots,x_{k+1}).$$

Let's expand F to a Fourier series (as an L^2 function on $\mathbb{T}^{k+2}$):

$$F(x_1,\dots,x_{k+1}) = \sum_{\underline{m}\in\mathbb{Z}^{k+1}} \hat{F}(\underline{m})\, e^{2\pi i\langle \underline{m},\underline{x}\rangle}, \text{ where}$$

$$\hat{F}(\underline{m}) = \int_0^1 \cdots \int_0^1 e^{-2\pi i\langle \underline{m},\underline{x}\rangle} F(\underline{x})\,dx_1 \cdots dx_{k+1}.$$

If we can show that $\hat{F}(\underline{m})$ for all $\underline{m} \neq \underline{0}$, we'll get that F, whence f, is constant (and equal a.e to $\hat{F}(\underline{0})$).

By the invariance of f;

$$F(x_1,\dots,x_{k+1}) = (f \circ T_{\beta,k+1})(\beta; x_1,\dots,x_{k+1}) = F(x_1+\beta, x_2+x_1,\dots,x_{k+1}+x_k)$$

$$= \sum_{\underline{m}\in\mathbb{Z}^{k+1}} \hat{F}(\underline{m}) \exp\left[2\pi i \left\langle \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_{k+1} \end{pmatrix}, \begin{pmatrix} x_1+\beta \\ x_2+x_1 \\ \vdots \\ x_{k+1}+x_k \end{pmatrix} \right\rangle \right]$$

$$= \sum_{\underline{m}\in\mathbb{Z}^{k+1}} \hat{F}(\underline{m})\, e^{2\pi i m_1 \beta}\, e^{2\pi i \left\langle \underline{m}, \begin{pmatrix} x_1 \\ x_2+x_1 \\ \vdots \\ x_{k+1}+x_k \end{pmatrix} \right\rangle}$$

$$= \sum_{\underline{m}\in\mathbb{Z}^{k+1}} \hat{F}(\underline{m})\, e^{2\pi i m_1 \beta}\, e^{2\pi i \langle \underline{m}, A\underline{x} \rangle}, \quad A = A_{k+1} = \underbrace{\begin{pmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ & 1 & 1 & & \\ 0 & & \ddots & \ddots & \\ & & & 1 & 1 \end{pmatrix}}_{k+1}$$

$$= \sum_{\underline{m}\in\mathbb{Z}^{k+1}} \hat{F}(\underline{m})\, e^{2\pi i m_1 \beta}\, e^{2\pi i \langle A^t \underline{m}, \underline{x} \rangle}$$

We now change the index of summation $\underline{m} = (A^t)^{-1}\underline{n}$, noting that $(A^t)^{-1}$ maps $\mathbb{Z}^{k+1}$ <u>bijectively</u> to $\mathbb{Z}^{k+1}$ (because it is an integer matrix with determinant one). This leads to

$$F(x_1,\dots,x_{k+1}) = \sum_{\underline{n}\in\mathbb{Z}^{k+1}} \hat{F}((A^t)^{-1}\underline{n})\, e^{2\pi i \beta ((A^t)^{-1}\underline{n})_1}\, e^{2\pi i \langle \underline{n}, \underline{x} \rangle}$$

Here $((A^t)^{-1}\underline{n})_1 = 1^{st}$ coord of $(A^t)^{-1}\underline{n} = n_1 - n_2 + n_3 - n_4 \pm \dots \pm n_{k+1}$,

because $\underline{m} = (A^t)^{-1}\underline{n} \iff (A^t)\underline{m} = \underline{n} \iff \begin{pmatrix} 1 & 1 & & 0 \\ & 1 & 1 & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_{k+1} \end{pmatrix} = \begin{pmatrix} n_1 \\ \vdots \\ n_{k+1} \end{pmatrix}$

$$\iff \begin{cases} m_1 + m_2 = n_1 \\ m_2 + m_3 = n_2 \\ \dots \\ m_k + m_{k+1} = n_k \\ m_{k+1} = n_{k+1} \end{cases} \Rightarrow \boxed{m_1 = n_1 - n_2 + n_3 \mp \dots n_{k+1}}$$

Since the Fourier expansion is unique, we can equate coefficients and find that

$$\boxed{\forall \underline{n} \in \mathbb{Z}^{k+1} \quad \hat{F}(\underline{n}) = \hat{F}((A^t)^{-1}\underline{n}) \cdot e^{2\pi i \beta \sum_{j=1}^{k+1} (-1)^{j+1} n_j} .}$$

Passing to absolute values leads to

$$\forall \underline{m} \in \mathbb{Z}^{k+1} \quad |\hat{F}(\underline{m})| = |\hat{F}((A^t)^{-1}\underline{m})|$$

or equivalently, to

$$\boxed{\forall \underline{m} \in \mathbb{Z}^{k+1} \quad |\hat{F}(\underline{m})| = |\hat{F}(A^t \underline{m})|}$$

Recall Parseval's identity: $\sum_{\underline{m}\in\mathbb{Z}^d} |\hat{F}(\underline{m})|^2 = \int_{\mathbb{T}^{k+2}} F^2\, dm$

Since F is bounded, $\sum_{\underline{m}\in\mathbb{Z}^d} |\hat{F}(\underline{m})|^2 < \infty$.

Necessarily:

$$\hat{F}(\underline{m}) \neq 0 \Rightarrow \{(A^t)^n \underline{m} : n\in\mathbb{Z}\} \text{ is finite.}$$

$$\Rightarrow \exists\, p>q \text{ s.t. } (A^t)^p \underline{m} = (A^t)^q \underline{m}$$

$$\underset{\det A\neq 0}{\Rightarrow} (A^t)^{p-q}\underline{m} = \underline{m} \Rightarrow \underline{m}^t A^{p-q} = \underline{m}$$

Claim: Necessarily $m_2 = m_3 = \dots = m_{k+1} = 0$

Proof. Let $\ell = p-q$. Recall that $A^\ell = (\alpha_{i-j})_{k+1\times k+1}$, $\alpha_{i-j} = \binom{\ell}{i-j}$.
The equation $\underline{m}^t A = \underline{m}^t$ is

$$(m_1, \dots, m_{k+1}) \begin{pmatrix} 1 & & & & & \\ \alpha_1 & 1 & & & 0 & \\ \alpha_2 & \alpha_1 & 1 & & & \\ \alpha_3 & \alpha_2 & \alpha_1 & \ddots & & \\ \vdots & \vdots & \vdots & & 1 & \\ \alpha_{k+1} & \alpha_k & \alpha_{k-1} & \cdots & \alpha_1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ m_1 \\ \\ \\ m_{k+1} \end{pmatrix}$$

Eqn $k+1$: $m_{k+1} = m_{k+1}$

Eqn k: $m_k + \alpha_1 m_{k+1} = m_k \Rightarrow m_{k+1} = 0$

Eqn $k-1$: $m_{k-1} + \alpha_1 m_k + \alpha_2 m_{k+1} = m_{k-1} \Rightarrow m_k = 0$

. . . .

Eqn 1: $m_1\alpha_1 +$ zeroes $= 0 \Rightarrow m_2 = 0$.

So $m_2 = m_3 = \dots = m_{k+1} = 0$ as claimed.

It remains to see that $\hat{F}(\underline{m}) \neq 0 \Rightarrow m_1 = 0$.

We just saw that $\hat{F}(\underline{m}) \neq 0 \Rightarrow m_2 = m_3 = \dots = m_{k+1} = 0$.

Substituting this in the equation

$$\forall \underline{m} \in \mathbb{Z}^{k+1} \quad \hat{F}(\underline{m}) = \hat{F}((A^t)^{-1}\underline{m}) \cdot e^{2\pi i \beta ((A^t)^{-1}\underline{m})_1}$$

Since the LHS $\neq 0$ and RHS $\neq 0$,

$$\underline{m} = \begin{pmatrix} m_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (A^t)^{-1}\underline{m} = \begin{pmatrix} [(A^t)^{-1}\underline{m}]_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

In addition, as we saw above,

$$((A^t)^{-1}\underline{m})_1 = m_1 \underbrace{- m_2 + m_2 \mp \dots \pm m_{k+1}}_{=0} = m_1$$

So

$$\hat{F}(\underbrace{m_1, 0, \dots, 0}_{\underline{m}}) = \hat{F}(\underbrace{m_1, 0, \dots, 0}_{(A^t)^{-1}\underline{m}})\, e^{2\pi i \beta m_1}.$$

Since we're assuming that $\hat{F}(\underline{m}) \neq 0$, we can divide by $\hat{F}(\underline{m})$ and obtain

$$e^{2\pi i \beta m_1} = 1 \Rightarrow \beta m_1 \in \mathbb{Z}.$$

But $\beta \notin \mathbb{Q}$. Necessarily $\boxed{m_1 = 0}$.

In summary, $\hat{F}(\underline{m}) \neq 0 \Rightarrow \underline{m} = 0$.

It follows that $f(\beta, x_1, \dots, x_d) = \hat{F}(\underline{0})\, e^{2\pi i \langle \underline{0}, \underline{x} \rangle} = \text{const}.$

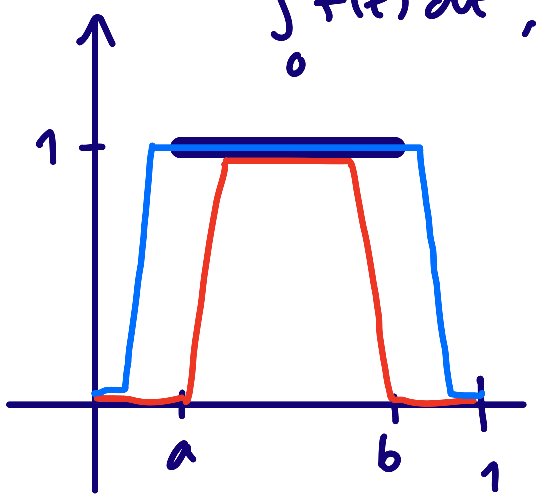
We just showed that any bounded measurable $T_{\beta,k+1}$- inv. function is constant almost everywhere. It follows that $T_{\beta,k+1}$ is ergodic, whence by Furstenberg's thm, uniquely ergodic.

Step 3. Uniform Distribution of $\{n^k\alpha\}$.

Fix some $(a,b) \subseteq (0,1)$. Given $\varepsilon>0$, build two continuous functions $f(t), g(t)$ on $\mathbb{T}$ s.t.

$$0 \le f(t) \le 1_{(a,b)}(t) \le g(t)$$

$$\int_0^1 f(t)\,dt,\ \int_0^1 g(t)\,dt \text{ are } \varepsilon\text{-close to } |a-b|$$



Next define $F, G : \beta\times\mathbb{T}^k \to \beta\times\mathbb{T}^k$ by

$$F(\beta; x_1, \dots, x_k) = f(x_k)$$
$$G(\beta; x_1, \dots, x_k) = g(x_k)$$

For the initial condition $p = (\beta; \xi_1, \dots, \xi_k)$ s.t. $T^n(p) = (\beta; * \cdots *, \{n^k\alpha\})$ we have

$$\frac{1}{N}\#\{1\le n\le N : \{n^k\alpha\}\in(a,b)\} \le \frac{1}{N}\sum_{n=1}^{N} g(\{n^k\alpha\})$$

$$= \frac{1}{N}\sum_{n=1}^{N} G(T^n p) \quad (\because T^n p = (* \cdots * \{n^k\alpha\}))$$

$$\xrightarrow[N\to\infty]{} \int_{\beta\times\mathbb{T}^k} G\,dm_k = \int_0^1 \cdots \int_0^1 f(x_k)\,dx_1 \cdots dx_k = \int_0^1 f(t)\,dt$$

$$\le |a-b| + \varepsilon$$

Thus $\limsup_{N\to\infty} \frac{1}{N} \#\{1\le n\le N : \{n^k\alpha\}\in(a,b)\} \le |a-b|+\varepsilon$.

Similarly, using f and F, we can show that

$$\liminf_{N\to\infty} \frac{1}{N} \#\{1\le n\le N : \{n^k\alpha\}\in(a,b)\} \ge |a-b|-\varepsilon.$$

Since ε is arbitrary, $\liminf = \limsup = |a-b|$. $\square$

This completes the proof of Weyl's Uniform Distribution theorem for $\{n^k\alpha\}$.

<u>Exercise</u>: Suppose $p(t) = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0$ and a_d is irrational. Prove that $\{p(n)\}$ is uniformly distributed in $[0,1]$. ($\{\cdot\}$ = fractional part).