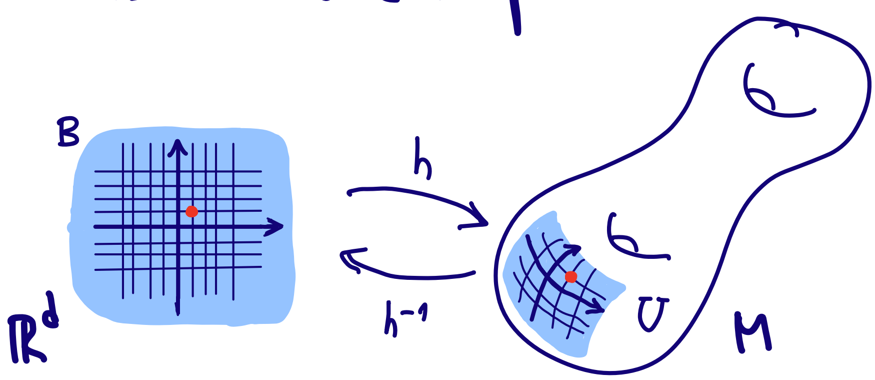
Review of Manifolds

Differentiable manifolds are spaces with the structure needed to do calculus in the space. The essence of the structure are "local coordinate systems" which allow to present functions on the space as functions on (open pieces of) $\mathbb{R}^d$.

Topological Manifolds: A *d-dimensional topological manifold* is a second countable Hausdorff topological space* (e.g. a compact metric space) s.t. for any point p, there's a *local coordinate system* (h, U, B):

- $U \subseteq M$ is an open neigh of p
- $B \subseteq \mathbb{R}^d$ is an open set homeomorphic to an open ball in $\mathbb{R}^d$
- $h: U \to B$ is a homeomorphism



---

* *Hausdorff*: Different points have disjoint neighborhood

Second Countable: $\exists$ countable collection of open sets U_i s.t. any open set is a union of U_i's.

Homeomorphic: A is homeomorphic to B if $\exists\, h: A \to B$ invertible s.t. h, h^{-1} are continuous. h is the *homeomorphism*.

Since $h: U \to B$ and $B \subseteq \mathbb{R}^d$, $h(q) = (x_1(q), \dots, x_d(q))$.
We call $(x_1(q), \dots, x_d(q))$ a <u>local coordinate system</u>.
Conversely, any d-tuple $(x_1, \dots, x_d) \in B$ determines a point $p(x_1, \dots, x_d) \in U \subseteq M$ defined by

$$p(x_1, \dots, x_d) = h^{-1}(x_1, \dots, x_d)$$

We call $p(\cdot, \dots, \cdot) = h^{-1}$ a <u>local chart</u>.

<u>"Writing in Coordinates"</u>:

- <u>Functions</u>: $\varphi: U \to \mathbb{R}$ becomes

$$\tilde{\varphi}(x_1, \dots, x_d) := (\varphi \circ h^{-1})(x_1, \dots, x_d) \qquad (\underline{x} \in B)$$

- <u>Curves</u>: $c: [-1, 1] \longrightarrow U$ becomes

$$\vec{c}(t) = (c \circ h^{-1})(t) = (x_1(t), \dots, x_d(t))$$

- <u>Maps</u>: $f: U_1 \longrightarrow U_2$ $\quad (U_i \subseteq M)$ becomes

$$\vec{f}(x_1, \dots, x_d) := h_2 \circ f \circ h_1^{-1}(x_1, \dots, x_d) = (y_1(\underline{x}), \dots, y_d(\underline{x}))$$

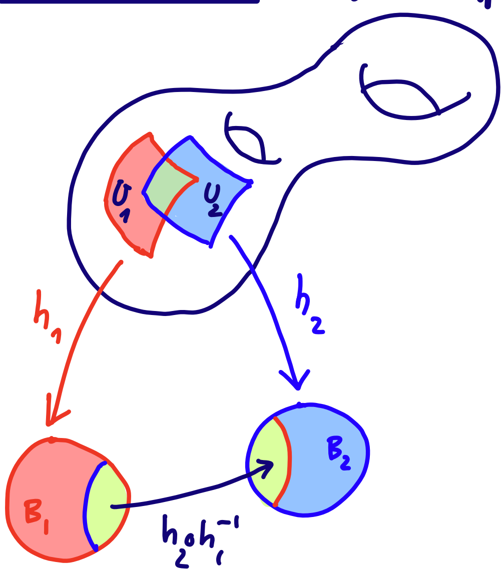
where $h_1: U_1 \to B_1$ is $h_1(q) = (x_1(q), \dots, x_d(q))$

$h_2: U_2 \to B_2$ is $h_2(q) = (y_1(q), \dots, y_d(q))$.

<u>The Main Issue</u>: There are <u>many</u> choices of local coordinate systems $h: U \to B$. We must be carefull to keep all our definitions independent of the choice of coordinates

C^k-Differentiable Manifold of Dimension d

A topological d-dimensional manifold M with system of charts (h_i, U_i, B_i) with the following <u>compatibility condition</u>: If $U_1 \cap U_2 \neq \emptyset$, then



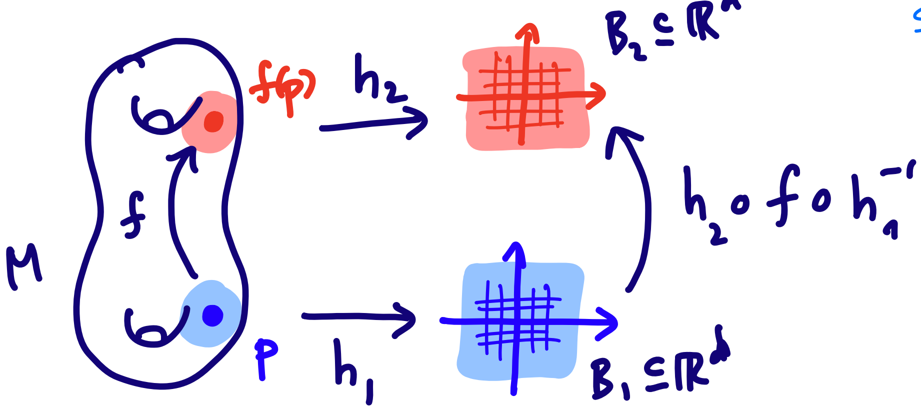
$$h_2 \circ h_1^{-1} : \underbrace{h_1(U_1 \cap U_2)}_{\subseteq \mathbb{R}^d} \longrightarrow \underbrace{h_2(U_1 \cap U_2)}_{\subseteq \mathbb{R}^d}$$

is continuously differentiable k times

- $\varphi : M \to \mathbb{R}$ is <u>differentiable</u> at $p \in M$ if for some (any) local coordinate chart (h, U, B), $U \ni p$, "φ in coordinates" $(\varphi \circ h^{-1})(x_1, \ldots, x_d)$ is differentiable on B.

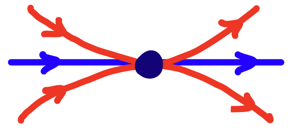
- $c : (-1,1) \to M$ is <u>differentiable</u> at 0 if for some (any) local coord system (h, U, B), $U \ni c(0)$, "c in coord" $(h \circ c)(t)$ is diff at $t = 0$.

- $f : M \to M$ is <u>differentiable</u> at $p \in M$, if for some (any) local coordinates (h_1, U_1, B_1), (h_2, U_2, B_2) s.t. $U_1 \ni p$, $U_2 \ni f(p)$ "f in coordinates" $h_2 \circ f \circ h_1^{-1} : \underbrace{B_1 \cap h_2^{-1}(U_2)}_{\subseteq \mathbb{R}^d} \longrightarrow \mathbb{R}^d$ is differentiable



The def^{n} of diff manifolds makes the def^{n} of differentiability independent of charts.

Tangent Space: Intuitively, the tangent vectors at $p \in M$ represent all possible directions of infinitesimal motion at p.

"Directions of Motion" are $\dot{c}(0)$ for curves $c(t)$ s.t. $c(0)=p$.
But we need to identify tangent curves  :

Def$^{\underline{n}}$ Two differentiable curves $c_i : (-1,1) \to M$ s.t. $c_1(0) = c_2(0) = p$ are tangent at p, if for some (any) local coordinate chart (h, U, B),

$$\frac{d}{dt}\Big|_{t=0} h\,(c_1(t)) = \frac{d}{dt}\Big|_{t=0} h\,(c_2(t))$$

This is an equivalence relation. Denote the equiv classes by $\dot{c}(0)$.

Def$^{\underline{n}}$. A tangent vector at p is an equivalence class of a differentiable curve $c : (-1,1) \to M$ s.t. $c(0) = p$.

Def$^{\underline{n}}$. The directional derivative at p in the direction of a tangent vector $\vec{v} := \dot{c}(0)$ is the operator acting on diff functions $\varphi : M \to \mathbb{R}$ by

$$D_{\vec{v}}(\varphi) = \frac{d}{dt}\Big|_{t=0} \varphi(c(t))$$

Exercise: $D_{\vec{v}}(\varphi)$ is independent of the choice of representative c of $\dot{c}(0)$.

Exercise: $D_{\vec{v}_1} = D_{\vec{v}_2} \implies \vec{v}_1 = \vec{v}_2$

Def$^{\underline{n}}$. Suppose $\vec{v}_1, \vec{v}_2$ are two tangent vectors at p, corresponding to $\dot{c}_1$, $\dot{c}_2$. Fix $\alpha, \beta \in \mathbb{R}$. We define

$$\alpha \vec{v}_1 + \beta \vec{v}_2$$

to be the unique tangent vector s.t. $D_{\alpha\vec{v}_1+\beta\vec{v}_2} = \alpha D_{\vec{v}_1} + \beta D_{\vec{v}_2}$.

Here is its construction: $v = \dot{c}_3(0)$ where:

- $(h \circ c_1)(t) = (x_1(t), \dots, x_d(t)) = \underline{x}(t)$
- $(h \circ c_2)(t) = (y_1(t), \dots, y_d(t)) = \underline{y}(t)$
- $c_3(t) :=$ curve through p s.t.

$$(h \circ c_3)(t) = \underbrace{\underline{x}(0)}_{\text{coord of } p} + \underbrace{\alpha(\underline{x}(t) - \underline{x}(0)) + \beta(\underline{y}(t) - \underline{y}(0))}_{\text{velocity} = \alpha\dot{x} + \beta\dot{y}}$$

Exercise: Check that this def$^{\underline{n}}$ is independent of the coordinate chart (i.e. the c_3 obtained from different charts are always tangent at $t=0$).

Exercise: Check that with this construction

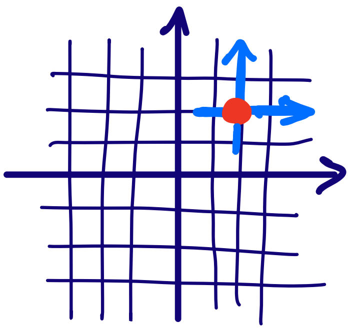
$$D_{\alpha\vec{v}_1+\beta\vec{v}_2} = \alpha D_{\vec{v}_1} + \beta D_{\vec{v}_2}$$

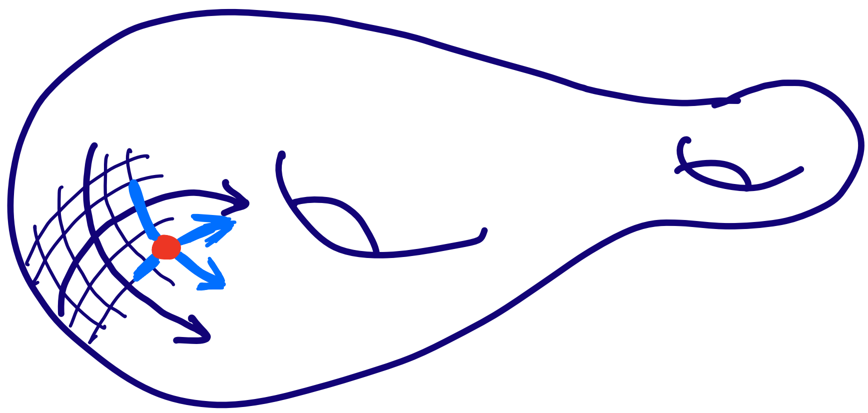
Def$^{\underline{n}}$. The vector space $T_pM = \{\text{tangent vectors at } p\}$ thus obtained is called the tangent space at p.

Thm. $\dim(T_pM) =$ dimension of the manifold. Moreover, if (h, U, B) is a coordinate system s.t. $U \ni p$, then

$$T_pM = \operatorname{Span}\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right\}$$

where $\partial/\partial x_i = \dot{c}_i(0)$, $c_i(t) = \underbrace{h^{-1}(h(p) + t\underline{e}_i)}_{\text{the curve with constant velocity } \underline{e}_i}$, $\underline{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← i.





Origin of Notation: If $f : M \to \mathbb{R}$ is given in coordinates by $\tilde{f}(x_1, \dots, x_d) = (f \circ h^{-1})(x_1, \dots, x_d)$ then

$$D_{\partial/\partial x_i} f = \frac{d}{dt}\Big|_{t=0} f(c_i(t)) = \frac{d}{dt}\Big|_{t=0} (f \circ h^{-1})(h(c_i(t))) =$$

$$= \frac{d}{dt}\Big|_{t=0} \tilde{f}(h(p) + t\underline{e}_i) = \frac{\partial \tilde{f}}{\partial x_i}(h(p))$$

Proof of Thm. Clearly $\partial/\partial x_1, \dots, \partial/\partial x_d \in T_pM$.

Fix some $c : (-1,1) \to M$ diff at zero s.t. $c(0) = p$.

Let $\tilde{c}(t) = (h \circ c)(t)$, a curve in $\mathbb{R}^d$.

Since c is diff at $t=0$, $\tilde{c}$ is diff at $t=0$. Write

$$\frac{d}{dt}\Big|_{t=0} \tilde{c}(t) = \begin{pmatrix} \tilde{c}_1 \\ \vdots \\ \tilde{c}_d \end{pmatrix}.$$

Let $\xi(t) = h^{-1}\left(h(p) + \sum_{i=1}^{d} \tilde{c}_i \underline{e}_i\right)$. This is a curve s.t. $\xi(0) = p$ and

$$\dot{\xi}(0) = \sum_{i=1}^{d} \tilde{c}_i \, \partial/\partial x_i .$$

At the same time,

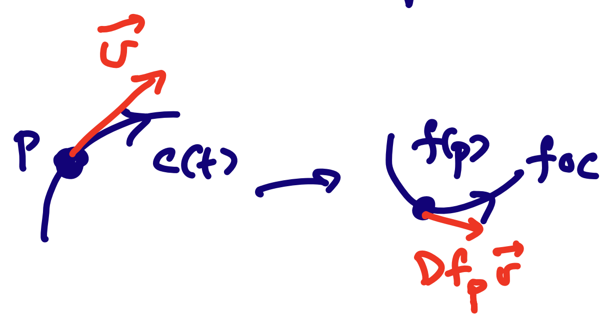
$$\frac{d}{dt}\Big|_{t=0} (h \circ \xi)(t) = \begin{pmatrix} \tilde{c}_1 \\ \vdots \\ \tilde{c}_d \end{pmatrix} = \frac{d}{dt}\Big|_{t=0} (h \circ c)(t)$$

So $\dot{\xi}(0) = \dot{c}(0)$. Thus $\dot{c}(0) = \sum_{i=1}^{d} \tilde{c}_i \, \partial/\partial x_i \in \operatorname{Span}\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right\}$. □

The Differential Suppose $f: M \to M$ is a differentiable map. The **differential of f at p** is the linear map

$$Df_p : T_pM \to T_{f(p)}M$$

$$(Df_p)(\dot{c}(0)) = (\dot{f \circ c})(0)$$



Exercise: Check that this is independent of the choice of c in $\dot{c}(0)$.

Exercise: Suppose (h_1, U_1, B_1), (h_2, U_2, B_2) are local coordinate charts s.t. $U_1 \ni p$, $U_2 \ni f(p)$. Write f in coordinates $(h_2 \circ f \circ h_1)(x_1, \dots, x_d) = (y_1(\underline{x}), \dots, y_d(\underline{x}))$.

Suppose we use the bases

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \quad \text{for } T_pM \quad (\text{given by } (h_1, U_1, B_1))$$

$$\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_d} \quad \text{for } T_{f(p)}M \quad (\text{given by } (h_2, U_2, B_2))$$

and we write

$$(Df_p)\left(\sum_{i=1}^{d} \alpha_i \frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{d} \beta_j \frac{\partial}{\partial y_j}.$$

Then

$$\underline{\beta} = \left(\frac{\partial y_i}{\partial x_j}\right)_{d \times d} \underline{\alpha}.$$

Riemannian Manifolds

Def^n. Let M be a d-dimensional differentiable manifold M. A Riemannian metric on M is an assignment of inner products $\langle \cdot, \cdot \rangle_p$ to T_pM in such a way that for some (any) local coordinate system (h, U, B), the functions

$$g_{ij}(p) := \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_p$$

are smooth. The associated Riemannian norm is

$$\|\vec{v}\|_p := \langle \vec{v}, \vec{v} \rangle_p.$$

We define the length of a differentiable curve $\gamma:(a,b)\to M$ by $L(\gamma)=\int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)}\,dt$.

The Riemannian distance between $p,q\in M$ is the infimum of the lengths of all smooth curves γ from a to b.

(If there are no such curves, the distance is $+\infty$.)