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Unique Ergodicity for Infinite Measures

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Abstract

We survey examples of dynamical systems on non–compact spaces which exhibit measure rigidity on the level of infinite invariant measures in one or more of the following ways: all locally finite ergodic invariant measures can be described; exactly one (up to scaling) admits a generalized law of large numbers; the generic points can be specified. The examples are horocycle flows on hyperbolic surfaces of infinite genus, and certain skew products over irrational rotations and adic transformations. In all cases, the locally finite ergodic invariant measures are Maharam measures.

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1. Introduction

1.1. Motivation. A continuous map T on a compact metric space Ω_0 is called uniquely ergodic if it has exactly one invariant probability measure. It is natural to ask what is the right notion of "unique ergodicity" for maps on non–compact spaces whose invariant measures are all infinite. The question is not what is possible, but rather what happens for "natural" examples.

Following a program initiated in [ANSS], we studied the measure rigidity of non–compact analogues of classical uniquely ergodic systems. The systems we studied include horocycle flows on surfaces of infinite genus, and non–compact group extensions of irrational rotations and adic transformations. The purpose of this lecture is to present our findings, and indicate some open problems.

This work grew out of the vision of Jon Aaronson, and it is with great pleasure that I dedidate this paper to him, on the occasion of his birthday.

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1.2. Basic Definitions. Let T be a measurable map on a measurable space $(\Omega, \mathscr{B})$, and suppose μ is a σ–finite measure on $(\Omega, \mathscr{B})$ s.t. $\mu(\Omega) \neq 0$. We say that μ is *invariant*, if $\mu(T^{-1}E) = \mu(E)$ for all $E \in \mathscr{B}$. We say that μ is *ergodic*, if for every set $E \in \mathscr{B}$ s.t. $T^{-1}(E) = E$, either $\mu(E) = 0$ or $\mu(\Omega \setminus E) = 0$.

We say that μ is *conservative*, if for every $W \in \mathscr{B}$ s.t. $\{T^{-n}(W)\}_{n \geq 0}$ are pairwise disjoint, $\mu(W) = 0$. This condition is always satisfied in the following cases: (1) μ is a finite invariant measure; and (2) μ is σ–finite non–atomic ergodic invariant measure and T is a bimeasurable bijection [A1].

The ergodic theorems describe the information such measures contain on the almost sure behavior of orbits $\{T^k\omega\}_{k \geq 0}$ ($T^k := T \circ \cdots \circ T$, k times):

Theorem 1.1 (Birkhoff). *Let μ be a finite ergodic invariant measure for $T : \Omega \to \Omega$, then for every $f \in L^1(\Omega, \mathscr{B}, \mu)$, $\frac{1}{N}\sum_{k=1}^{N} f(T^k\omega) \xrightarrow[N\to\infty]{} \frac{1}{\mu(\Omega)}\int_\Omega f d\mu$ μ–a.e.*

Theorem 1.2 (Hopf). *Suppose μ is a σ–finite conservative ergodic invariant measure, then for every $f, g \in L^1(\Omega, \mathscr{B}, \mu)$ s.t. $g \geq 0$ and $\int_\Omega g d\mu > 0$,*

$$\frac{\sum_{k=1}^{N} f(T^k\omega)}{\sum_{k=1}^{N} g(T^k\omega)} \xrightarrow[N\to\infty]{} \frac{\int_\Omega f d\mu}{\int_\Omega g d\mu} \quad \text{for } \mu\text{–almost every } \omega.$$

Specializing to the case when f and g are indicator functions of sets F, G of positive finite measure, we see that if $\mu(\Omega) = \infty$ then the frequency of visits of $T^n(\omega)$ to F and G tends to zero, but the ratio of these frequencies tends to a definite limit.

The limit depends on μ, although it is the same for proportional measures. It is therefore of great interest to know what are the possible ergodic invariant measures up to scaling. To avoid pathologies (cf. [Sch2]), we restrict our attention to measures which are locally finite in some sense which we now make precise.

The following set-up is not the most general possible, but suffices for our purposes. Suppose Ω_0 is a locally compact second countable metric space with Borel σ–algebra $\mathscr{B}_0$. Let $C_c(\Omega_0) := \{f : \Omega_0 \to \mathbb{R} : f$ continuous with compact support$\}$. A Borel measure μ on Ω_0 is called a *Radon measure*, if $\mu(C) < \infty$ for every compact set $C \subset \Omega_0$. Equivalently, every $f \in C_c(\Omega_0)$ is absolutely integrable.

In §14 we will need to deal with Borel maps T which are only defined on a subset $\Omega \subseteq \Omega_0$, $\Omega \in \mathscr{B}_0$. Let $\mathscr{B} := \{E \cap \Omega : E \in \mathscr{B}_0\}$. A measure μ on $(\Omega, \mathscr{B})$ is called *locally finite*, if $\mu_0(E) := \mu(E \cap \Omega)$ is a Radon measure on Ω_0. If $\Omega = \Omega_0$, then the properties of being Radon and being locally finite are the same.

Theorems 1.1 and 1.2 are almost sure statements. It is interesting to know what are their points of validity.

Definition 1.1. *A point $\omega \in \Omega$ is called* generic *for μ if*

1. *$\mu(\Omega) < \infty$ and for all $f \in C_c(\Omega_0)$, $\frac{1}{N}\sum_{k=1}^{N} f(T^k\omega) \xrightarrow[N\to\infty]{} \frac{1}{\mu(\Omega)}\int_\Omega f d\mu$; or*

2. *$\mu(\Omega) = \infty$, and for all $f, g \in C_c(\Omega_0)$ such that $g \geq 0$ and $\int g d\mu > 0$,*

$$\frac{\sum_{k=1}^{N} f(T^k\omega)}{\sum_{k=1}^{N} g(T^k\omega)} \xrightarrow[N\to\infty]{} \frac{\int_\Omega f d\mu}{\int_\Omega g d\mu}.$$

Our assumptions on Ω_0 and Hopf's theorem guarantee that the set of generic points of a locally finite conservative ergodic invariant measure μ has full μ–measure.

Similar definitions can be made for flows. A *Borel flow* $\varphi : \Omega \to \Omega$ is a group of maps $\varphi^t : \Omega \to \Omega$ ($t \in \mathbb{R}$) such that $(t, \omega) \mapsto \varphi^t(\omega)$ is Borel, and $\varphi^t \circ \varphi^s = \varphi^{t+s}$ ($t, s \in \mathbb{R}$). A Borel measure is called *φ–invariant*, if it is φ^t–invariant for all t. A Borel measure is called *φ–ergodic*, if any Borel set E s.t. $\varphi^{-t}(E) = E$ for all t satisfies $\mu(E) = 0$ or $\mu(\Omega \setminus E) = 0$. A point is called *generic* for a flow, if it satisfies definition 1.1 with $\int_0^N h(\varphi^s\omega)ds$ replacing $\sum_{k=1}^{N} h(T^k\omega)$ $(h = f, g)$.

1.3. Measure Rigidity.

Let T be a Borel map on a Borel subset Ω of a second countable locally compact metric space Ω_0. We are interested in the following problems:

1. Find all locally finite T–ergodic invariant measures;
2. Describe their generic points;
3. If there are many measures, find an ergodic theoretic property which singles out just one (up to scaling).

If one or more of these questions can be answered, then we speak (somewhat unorthodoxly) of "measurable rigidity". The strongest form of measure rigidity is unique ergodicity:

Definition 1.2. *T is* uniquely ergodic (u.e.) *if (1) T admits one locally finite invariant measure up to scaling; and (2) every point is generic for this measure.*[1]

It is useful to weaken this as follows. Let δ_y denote the point mass at y. A point ω is called *exceptional* for a map T (resp. a flow φ) if the measure $\sum_{n>0} \delta_{T^n(\omega)}$ (resp. $\int_0^\infty \delta_{\varphi^s(\omega)}ds$) is locally finite.

Definition 1.3. *T is* uniquely ergodic in the broad sense *if (1) up to scaling, T admits one locally finite ergodic invariant measure not supported on a*

[1] Usually unique ergodicity is only defined for continuous maps on compact metric spaces. In this case the unique invariant measure is finite, and (1) implies (2).

single orbit; and (2) every non–exceptional non–periodic point is generic for this measure.

See theorems 2.3 and 2.5 for examples.

Interestingly enough, in the non–compact case there is a large collection of "natural" examples which exhibit a different, more subtle, form of measure rigidity. For these dynamical systems:

- There are no finite invariant measures at all, except perhaps measures supported on periodic orbits;
- There are infinitely many locally finite ergodic invariant measures, all of which can be specified;
- In some cases we know what are the generic points of these measures;
- In some cases we are able show that exactly one of these measures up to scaling admits a generalized law of large numbers (cf. §10).

The purpose of this paper is to describe these examples.

2. Horocycle Flows

2.1. Definition. Let M be a complete, connected, orientable hyperbolic surface. Let T^1M be its unit tangent bundle. The *geodesic flow* is the flow $g : T^1M \to T^1M$ which moves a unit tangent vector, at unit speed, along its geodesic. The *Horocycle* of a vector $\omega \in T^1M$ is the set

$$\mathrm{Hor}(\omega) := \{\omega' \in T^1M : \mathrm{dist}(g^s\omega, g^s\omega') \xrightarrow[s\to\infty]{} 0\}. \tag{2.1}$$

We shall soon see that this is a smooth curve in T^1M. The *horocycle flow* of M is the flow $h : T^1M \to T^1M$ which moves $\omega \in T^1M$ at unit speed along $\mathrm{Hor}(\omega)$ in the positive direction.[2]

It is useful to consider the case when $M = \mathbb{H} := \{x + iy : x, y \in \mathbb{R}, y > 0\}$, equipped with the metric $\sqrt{dx^2 + dy^2}/y$. Poincaré's Theorem ([Kat], chapter 1), says that the orientation preserving isometries of $\mathbb{H}$ are Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ where a, b, c, d are real. We denote the collection of these maps by Möb($\mathbb{H}$). Möb($\mathbb{H}$) acts transitively on $T^1\mathbb{H}$: for every $\omega_1, \omega_2 \in T^1\mathbb{H}$ there exists $\varphi \in$ Möb($\mathbb{H}$) s.t. $\varphi_*(\omega_1) = \omega_2$. Schwarz's Lemma says that φ is unique.

Let ω_0 denote the unit tangent vector based at i and pointing north. It is easy to see that the geodesic flow moves ω_0 along the vertical ray it determines. Since every $\omega \in T^1\mathbb{H}$ can be mapped by an element of Möb($\mathbb{H}$) to ω_0, and since

[2] Sometimes h is called the *stable* horocycle flow. The unstable horocycle flow is defined in the same way, except that one takes the limit $s \to -\infty$ in (2.1).

isometries map geodesics to geodesics, every geodesic is either a circular arc perpendicular to $\partial\mathbb{H} := \{z : \operatorname{Im} z = 0\}$, or a vertical line.

One can check in a similar way that $\operatorname{Hor}(\omega_0)$ consists of the unit tangent vectors based on the line $\operatorname{Im} z = 1$ and pointing north. Since the hyperbolic metric agrees with the euclidean metric on the line $\operatorname{Im} z = 1$,

$$h^t(\omega_0) = (\psi_t)_*\omega_0, \text{ where } \psi_t : z \mapsto z + t.$$

For general vectors $\omega \in T^1\mathbb{H}$, let φ_ω be the unique element of $\text{Möb}(\mathbb{H})$ s.t. $\omega = (\varphi_\omega)_*\omega_0$, then $\operatorname{Hor}(\omega) = (\varphi_\omega)_*[\operatorname{Hor}(\omega_0)]$ and

$$h^t(\omega) = (\varphi_\omega \circ \psi_t)_*\omega_0. \tag{2.2}$$

The Möbius transformation φ_ω maps the line $\operatorname{Im} z = 1$ onto a circle C which is tangent to $\partial\mathbb{H}$ (possibly at ∞). $\operatorname{Hor}(\omega)$ consists of the unit tangent vectors based at C, perperndicular to C, and pointing in the direction of the tangency point.

There is a useful algebraic description of h. The elements of $\text{Möb}(\mathbb{H})$ are parametrized by the elements of

$$\mathrm{PSL}(2,\mathbb{R}) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R}, ad - bc = 1\right\} / \left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

We see that the map $\omega \mapsto$ coefficient matrix of φ_ω is bijection $T^1\mathbb{H} \to \mathrm{PSL}(2,\mathbb{R})$. Applying this identification to (2.2), we obtain a conjugacy between the horocycle flow on $T^1\mathbb{H}$ and the matrix flow $h : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$

$$h^t : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

This extends to other hyperbolic surfaces. The Killing–Hopf Theorem says that any complete orientable connected hyperbolic surface M is isometric to an orbit space $\Gamma \setminus \mathbb{H}$, where Γ is a discrete subgroup of $\text{Möb}(\mathbb{H})$ without elements of finite order ("torsion free"). Γ is called a *uniform lattice* if $\Gamma \setminus \mathbb{H}$ is compact, a *lattice* if $\Gamma \setminus \mathbb{H}$ has finite area, and *geometrically finite* if $\Gamma \setminus \mathbb{H}$ has finite genus. Every uniform lattice is a lattice, and every lattice is geometrically finite [Kat].

The identifications $T^1\mathbb{H} \simeq \text{Möb}(\mathbb{H}) \simeq \mathrm{PSL}(2,\mathbb{R})$ turn the horocycle flow on T^1M into the matrix flow $h : \Gamma \setminus \mathrm{PSL}(2,\mathbb{R}) \to \Gamma \setminus \mathrm{PSL}(2,\mathbb{R})$

$$h^t : \Gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \Gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Let m_0 denote the Riemannian volume measure on T^1M. We can use the algebraic representation of h to relate m_0 to the Haar measure of $\mathrm{PSL}(2,\mathbb{R})$, and to deduce its h–invariance. It is enough to treat the case $M = \mathbb{H}$, the general case follows from the representation $M = \Gamma \setminus \mathbb{H}$. The identification $\omega \mapsto$the coefficient matrix of φ_ω conjugates the action of $\text{Möb}(\mathbb{H})$ on $T^1\mathbb{H}$ to the action of PSL$(2,\mathbb{R})$ on itself by multiplication on the left. Isometries preserve volume, so m_0 must be mapped to the left Haar measure on PSL$(2,\mathbb{R})$. PSL$(2,\mathbb{R})$ is unimodular: its left Haar measure is invariant under multiplication on the right. Since h acts by multiplication on the right, m_0 is h–invariant.

Theorem 2.1 (Kaimanovich). *m_0 is h–ergodic iff every bounded harmonic function on M is constant ("Liouville property").*

This is in [Kai] (see also [Su], part II).

2.2. Horocycle flows on hyperbolic surfaces with finite genus.

Henceforth, unless stated otherwise, a "hyperbolic surface" means $\Gamma \setminus \mathbb{H}$, where Γ is a discrete torsion free subgroup of Möb$(\mathbb{H})$.

Recall the following chain of inclusions for hyperbolic surfaces [Kat]: compact $\subset$ finite area $\subset$ finite genus. The study of measure rigidity for horocycle flows starts with the following fundamental result [F1]:

Theorem 2.2 (Furstenberg). *If M is compact, then $h : T^1M \to T^1M$ is uniquely ergodic. The invariant measure is, up to scaling, m_0.*

A non–compact hyperbolic surface of finite area has "cusps" (Fig. 1a): pieces which are isometric to $C := \langle z \mapsto z+1 \rangle \setminus \{z \in \mathbb{H} : \operatorname{Im} z \geq a\}$ (where $a > 0$). Cusps contain periodic horocycles. In fact any unit tangent vector $\omega \in T^1C$ which points north is h–periodic, and the Lebesgue measure on its orbit is a finite invariant measure. It follows that the horocycle flow is not uniquely ergodic. But it is uniquely ergodic in the broad sense:

Theorem 2.3 (Dani–Smillie). *Suppose M is a hyperbolic surface of finite area.*

1. *The ergodic invariant Radon measures are up to scaling the volume measure m_0, and the measures supported on periodic horocycles.*
2. *Every $\omega \in T^1M$ whose horocycle is not periodic is generic for m_0.*

Part 1 is in [Da], part 2 is in [DS].

We see that in the finite area case all invariant measures are finite. For infinite area surfaces there are no finite invariant measures at all, other than measures supported on periodic horocycles (Ratner [Rat1]). We discuss the finite genus case. To avoid trivial exceptions we always assume that the area is infinite, and we only consider non–elementary surfaces, i.e. surfaces $M = \Gamma \setminus \mathbb{H}$ for which Γ is not generated by a single element.

Such surfaces have "funnels". These are subsets which are isometric to $F := \langle z \mapsto \lambda z \rangle \setminus \{z \in \mathbb{H} : \operatorname{Re} z \geq 0\}$, where $\lambda > 1$ (Fig. 1a). Funnels contain exceptional orbits: if the geodesic of $\omega \in T^1F$ tends to some $p \in \{z \in \partial\mathbb{H} : \operatorname{Re} z > 0\}$, then $\int_0^\infty \delta_{h^t\omega} dt$ is a Radon measure. The Radon property is because the horocycle eventually enters one fundamental domain of $\langle z \mapsto \lambda z \rangle$ and stays there without accumulating anywhere.

The set of exceptional ω's constructed above is an h invariant set of positive volume. Its complement also has positive volume. It follows that m_0 is not ergodic.

There does exist an h–ergodic invariant Radon measure μ which gives any single orbit measure zero [Bu]. We describe it.

It is convenient to work in the unit disc model $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ together with the metric $2\sqrt{dx^2 + dy^2}/(1 - x^2 - y^2)$. The map $\vartheta : \mathbb{H} \to \mathbb{D}$, $\vartheta(z) = \frac{i-z}{i+z}$ is an isometry from $\mathbb{H}$ to $\mathbb{D}$. It can be used to represent M in the form $\Gamma_{\mathbb{D}} \setminus \text{Möb}(\mathbb{D})$, where $\Gamma_{\mathbb{D}} = \vartheta\Gamma\vartheta^{-1}$. We abuse notation and write $\Gamma = \Gamma_{\mathbb{D}}$.

$T^1(\mathbb{D})$ can be identified with $\partial\mathbb{D} \times \mathbb{R} \times \mathbb{R}$ via $(e^{i\theta}, s, t) \leftrightarrow (h^t \circ g^s)(\omega(e^{i\theta}))$, where h is the horocycle flow, g is the geodesic flow, and $\omega(e^{i\theta})$ is the element of $T^1(\mathbb{D})$ based at the origin, and pointing at $e^{i\theta}$. (These are "KAN–coordinates" for $T^1(\mathbb{D}) \cong T^1\mathbb{H} \cong \text{PSL}(2, \mathbb{R})$.) In these coordinates, Γ acts by

$$\varphi_* : (e^{i\theta}, s, t) \mapsto (\varphi(e^{i\theta}), s - \log|\varphi'(e^{i\theta})|, t + a(\varphi, e^{i\theta}, s)) \quad (\varphi \in \Gamma) \tag{2.3}$$

where $a(\varphi, e^{i\theta}, s)$ is some function which does not depend on t. The horocycle flow is just the linear translation on the t–coordinate.

We continue to assume that $M = \Gamma \setminus \mathbb{D}$ is non–elementary, and let $\Lambda(\Gamma)$ denote the *limit set* of Γ, equal by definition to $\partial\mathbb{D} \cap \overline{\{\Gamma z\}}$ for some (hence all [Kat]) $z \in \mathbb{D}$. Let $\delta(\Gamma)$ denote the *critical exponent* of Γ, equal by definition to the infimum of all δ s.t. $\sum_{\varphi\in\Gamma} \exp[-\delta \,\text{dist}(z, \varphi(z))] < \infty$. The following is in [Pat]:

Theorem 2.4 (Patterson). *There exists a probability measure ν on $\Lambda(\Gamma) \subseteq \partial\mathbb{D}$ such that $\frac{d\nu\circ\varphi}{d\nu} = |\varphi'|^{\delta(\Gamma)}$ for all $\varphi \in \Gamma$.*

One can now use (2.3) to verify by direct calculation that

$$d\mu(e^{i\theta}, s, t) := e^{\delta(\Gamma)s} d\nu(e^{i\theta}) ds dt \tag{2.4}$$

is a Γ–invariant h–invariant measure on $T^1\mathbb{D}$. Γ–invariance means that μ descends to an h–invariant Radon measure on T^1M. We call the resulting measure the *Burger measure*. It is an infinite Radon measure. The following theorem implies, through the ergodic decomposition, that it is ergodic.

Theorem 2.5 (Burger − Roblin). *Suppose $M = \Gamma \setminus \mathbb{H}$ is a non–elementary hyperbolic surface with finite genus and infinite area. The h–ergodic invariant Radon measures are up to scaling*

1. *The Burger measure;*

2. *Infinite measures carried by horocycles of unit tangent vectors whose forward geodesics escape to infinity through a funnel;*

3. *Finite measures carried by periodic horocycles whose forward geodesics escape to infinity through a cusp.*

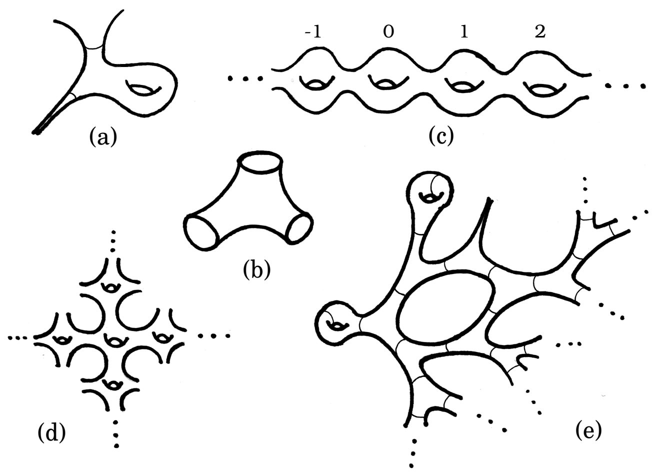

Figure 1. (a) A cusp, a funnel, and a handle; (b) A "pair of pants"; (c) A $\mathbb{Z}$–cover with its $\mathbb{Z}$–coordinates; (d) An F_2–cover of a compact surface; (e) A pants decomposition of a tame surface

The theorem was proved by Burger under the additional assumption that $\delta(\Gamma) > \frac{1}{2}$ and that M has no cusps [Bu]. The general case was done by Roblin [Ro], who also discusses extensions to variable negative curvature.

Theorem 2.6 (Schapira). *Suppose M is a hyperbolic surface of finite genus and infinite area, and let $\omega \in T^1M$. Either ω is h–periodic, or ω is exceptional, or ω is generic for the Burger measure.*

For a characterization of the generic $\omega \in T^1M$ in terms of the endpoints of their geodesics, see [Scha1], [Scha2]. For other equidistribution results which involve Burger's measure, see [Oh].

Together, theorems 2.5 and 2.6 say that the horocycle flow on a complete connected orientable hyperbolic surface of finite genus is uniquely ergodic in the broad sense.

2.3. Invariant measures in infinite genus.

Horocycle flows on hyperbolic surfaces of infinite genus are not always uniquely ergodic in the broad sense, as was first discovered by Babillot and Ledrappier.

Their examples are $\mathbb{Z}^d$–covers of compact hyperbolic surfaces [BL] (Fig. 1c). These are the surfaces of the form $M = \Gamma\backslash\mathbb{H}$, where Γ is a normal subgroup of a uniform lattice Γ_0 s.t. $\Gamma_0/\Gamma \simeq \mathbb{Z}^d$. Topologically, M is a regular cover of the compact surface $M_0 = \Gamma_0\backslash\mathbb{D}$, with covering map $p(\Gamma g) = \Gamma_0 g$. The covering group

$$\mathrm{Cov}(p) := \{D : M \to M : D \text{ is a homeomorphism s.t. } p \circ D = p\}$$

is isomorphic to $\mathbb{Z}^d$.

The elements of Cov(p) are called "deck transformations". They are isometries, and they take the form $\Gamma z \mapsto \Gamma g_0 z$ ($g_0 \in \Gamma_0$). We parametrize the deck transformations by $D_{\underline{\xi}}$ ($\underline{\xi} \in \mathbb{Z}^d$) in such a way that $D_{\underline{\xi}+\underline{\eta}} = D_{\underline{\xi}} \circ D_{\underline{\eta}}$. The deck transformations act on $T^1 M$ by their differentials. Abusing notation, we denote this action again by $D_{\underline{\xi}}$.

Theorem 2.7 (Babillot & Ledrappier). *For each $\underline{a} \in \mathbb{R}^d$ there exists up to scaling a unique h–ergodic invariant Radon measure $m_{\underline{a}}$ s.t. $m_{\underline{a}} \circ D_{\underline{\xi}} = e^{\langle \underline{a}, \underline{\xi} \rangle} m_{\underline{a}}$ ($\underline{\xi} \in \mathbb{Z}^d$).*

The parameter $\underline{a} = \underline{0}$ corresponds to m_0, the measure induced by the Haar measure. The measures $m_{\underline{a}}$ with $\underline{a} \neq \underline{0}$ are singular. Each is infinite, globally supported, and quasi–invariant under the geodesic flow $g : T^1 M \to T^1 M$: $\exists c(\underline{a})$ s.t. $m_{\underline{a}} \circ g^s = e^{c(\underline{a})s} m$. For a related result on nilpotent regular covers of compact hyperbolic surfaces, see Babillot [Ba].

Theorem 2.8 (S.). *Every h–ergodic invariant Radon measure is proportional to $m_{\underline{a}}$ for some $\underline{a}$.*

See [Sa2]. Notice that although there is more than one non–trivial ergodic invariant Radon measure, the collection of these measures is still small enough to be completely described.

Babillot noticed a striking similarity between the list $\{m_{\underline{a}} : \underline{a} \in \mathbb{R}^d\}$, and the list of minimal positive eigenfunctions of the Laplacian on M [Ba]. Some definitions:

- The *hyperbolic Laplacian of* $\mathbb{H}$ is a second order differential operator on $C^2(\mathbb{H})$ s.t. $\Delta_{\mathbb{H}}(f \circ \varphi) = (\Delta_{\mathbb{H}} f) \circ \varphi$ for all $\varphi \in$ Möb$(\mathbb{H})$. This determines $\Delta_{\mathbb{H}}$ up to a constant. With a particular choice of constant, $\Delta_{\mathbb{H}} = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$.
- The *hyperbolic laplacian of* $M = \Gamma \backslash \mathbb{H}$ is $(\Delta_M f)(\Gamma z) := (\Delta_{\mathbb{H}} \widetilde{f})(z)$ where $\widetilde{f}(z) := f(\Gamma z)$. The definition is proper, because of the commutation relation between $\Delta_{\mathbb{H}}$ and Möb$(\mathbb{H})$.
- The *positive λ–eigenfunctions* of Δ_M are the positive $F \in C^2(M)$ for which $\Delta_M F = \lambda F$. (We allow infinite L^2 norm.) We say that F is *minimal*, if $\Delta_M G = \lambda G$, $0 \leq G \leq F \Rightarrow \exists c$ s.t. $G = cF$. The minimal positive λ–eigenfunctions are the extremal rays of the cone of positive λ–eigenfunctions.

The minimal positive eigenfunctions of the Laplacian on a $\mathbb{Z}^d$–cover of a compact hyperbolic surface can be parametrized, up to scaling, by $\{F_{\underline{a}} : \underline{a} \in \mathbb{R}^d\}$, where $F_{\underline{a}} \circ D_{\underline{\xi}} = e^{\langle \underline{a}, \underline{\xi} \rangle} F_{\underline{a}}$ ($\underline{\xi} \in \mathbb{Z}^d$) (see [LP] and references therein). The similarity with the list of ergodic invariant Radon measures is obvious.

Motivated by this observation and Sullivan's work on the geodesic flow, Babillot proposed a method for getting invariant Radon measures out of positive eigenfunctions, and conjectured that at least in some cases her method provides a bijection between the two collections. We describe Babillot's construction.

Again, it is convenient to represent $M = \Gamma \setminus \mathbb{D}$, where Γ is a discrete torsion free subgroup of $\text{Möb}(\mathbb{D})$. The hyperbolic laplacian on $\mathbb{D}$ is $\Delta_{\mathbb{D}} f := [\Delta_{\mathbb{H}}(f \circ \vartheta)] \circ \vartheta^{-1}$, where $\vartheta : \mathbb{H} \to \mathbb{D}$ is the isometry $z \mapsto \frac{i-z}{i+z}$. The reader can check that $\Delta_{\mathbb{D}}$ commutes with $\text{Möb}(\mathbb{D})$, and that $\Delta_{\mathbb{D}} = \frac{1}{4}(1-|z|^2)^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$.

Any positive eigenfunction of Δ_M lifts to a Γ–invariant positive eigenfunction of $\Delta_{\mathbb{D}}$. Any positive eigenfunction of $\Delta_{\mathbb{D}}$ can be represented in the form

$$F(z) = \int_{\partial\mathbb{D}} P(e^{i\theta}, z)^{\alpha} d\nu_F(e^{i\theta}),$$

where ν is a finite positive measure on $\partial\mathbb{D}$, $P(e^{i\theta}, z) = (1-|z|^2)/|e^{i\theta} - z|^2$ is Poisson's kernel, and $\alpha \geq 1/2$ (Karpelevich [Kar], see also [GJT]). If $\delta(\Gamma) \geq \frac{1}{2}$, then this representation is unique, and the Γ–invariance of F translates to the following condition on ν:

$$\frac{d\nu_F \circ \varphi}{d\nu_F} = |\varphi'|^{\alpha} \text{ for all } \varphi \in \Gamma.$$

Comparing this with (2.3), we see that the measure

$$dm_F = e^{\alpha s} d\nu_F(e^{i\theta}) ds dt \tag{2.5}$$

is a Γ–invariant, h–invariant measure on $T^1(\mathbb{D})$. Its restriction to a fundamental domain of Γ induces an h–invariant measure on $M = \Gamma \setminus \mathbb{D}$.

Thus a positive eigenfunction F gives rise to a horocycle invariant Radon measure m_F. Babillot has conjectured – in the case of infinite regular covers of compact surfaces with nilpotent covering group – that every invariant Radon measure arises this way, and that minimal eigenfunctions F lead to ergodic invariant Radon measures m_F [Ba].

Babillot's conjecture was proved for all infinite regular covers in [LS2] (see [L] for a related result in higher dimension), and later for all *tame surfaces* [Sa1]. To explain what these are, we recall some definitions and facts [Hub]:

- A hyperbolic surface with boundary is called a *pair of pants*, if it is homeomorphic to a sphere minus three disjoint open discs or points (Fig. 1b).
- Every pair of pants has three boundary components of lengths $0 \leq \ell_i < \infty$ $(i = 1, 2, 3)$, where $\ell = 0$ corresponds to a cusp. Two pairs of pants with the same triplet of lengths are isometric.
- The *norm* of a pair of pants Y is the sum of the lengths of its boundary components, and is denoted by $\|Y\|$.
- A discrete subgroup $\Gamma \subset \text{Möb}(\mathbb{D})$ is called a *fuchsian group*. A fuchsian group is said to be *of the first kind* if its limit set $\Lambda(\Gamma)$ equals $\partial\mathbb{D}$.

- A torsion free fuchsian group Γ is of the first kind iff $\Gamma\backslash\mathbb{D}$ can be partitioned into a countable collection of pants $\{Y_i\}$ which meet at boundary components of the same length (see e.g. [Hub]).

We call $\{Y_i\}$ a *pants decomposition* of M.

Definition 2.1. *The surface* $\Gamma\setminus\mathbb{D}$ *is called* tame*, if it admits a pants decomposition* $\{Y_i\}$ *such that* $\sup\|Y_i\|<\infty$*.*

It can be shown that in this case $\delta(\Gamma)\geq\frac{1}{2}$ [Sa1].

Any regular cover of a compact hyperbolic surface is tame, because it admits an infinite pants decomposition whose components fall into finitely many isometry classes. There are many other examples: if one glues a finite or countable collection of pants of bounded norm one to another in such a way that every boundary component is glued to some other boundary component of the same length and orientation, then the result is a tame complete hyperbolic surface (Fig. 1e).

We need a couple more definitions to state the result.

- A horocycle ergodic invariant Radon measure is called *trivial* if it is supported on a single horocycle made of unit tangent vectors whose forward geodesics tend to a cusp.
- A Möbius function $\varphi\in\text{Möb}(\mathbb{D})$ is called *parabolic* if it has exactly one fixed point on $\partial\mathbb{D}$. A positive eigenfunction is called *trivial*, if it is of the form
$$F(z):=\sum_{g\in\Gamma/\text{stab}_\Gamma(e^{i\theta})}P(g(e^{i\theta}),z)^\alpha$$
where $P(\cdot,\cdot)$ is Poisson's kernel, $e^{i\theta}$ is a fixed point of some parabolic element of Γ, and $\text{stab}_\Gamma(e^{i\theta})=\{g\in\Gamma:g(e^{i\theta})=e^{i\theta}\}$.

The following theorem is proved in [Sa1], under slightly weaker assumptions.

Theorem 2.9 (S.). *If* $\Gamma\setminus\mathbb{D}$ *is tame, then the following map is a bijection between the non-trivial positive minimal eigenfunctions of the Laplacian on* $\Gamma\backslash\mathbb{D}$ *and the non-trivial horocycle ergodic invariant Radon measures on* $T^1(\Gamma\setminus\mathbb{D})$*:*

$$\left[F(\Gamma z)=\int_{\partial\mathbb{D}}P(e^{i\theta},z)^\alpha d\nu(e^{i\theta})\right]\leftrightarrow\left[\begin{array}{l}\textit{The restriction of } dm=e^{\alpha s}d\nu(e^{i\theta})dsdt\\ \textit{to a fundamental domain of } \Gamma\end{array}\right].$$

We illustrate the result by examples [LS2]:

1. *Furstenberg's Theorem:* All positive eigenfunctions on a compact surface are constant. The constant function maps to m_0. Consequently all ergodic invariant Radon measures are proportional to m_0.
2. *Dani's Theorem:* The minimal positive eigenfunctions on a hyperbolic surface of finite area are either constant, or trivial (Eisenstein series associated to cusps). So the ergodic invariant Radon measures are m_0 and trivial measures.

3. *Periodic surfaces of polynomial growth:* These are regular covers of compact hyperbolic surfaces with the property that the area of concentric balls of radius R is $O(R^\delta)$ for some δ as $R \to \infty$ (e.g. $\mathbb{Z}^d$–covers). Using Gromov's characterization of virtually nilpotent groups, it can be shown that the group of deck transformations contains a nilpotent normal subgroup N of finite index. The minimal positive eigenfunctions form a family
$$\{cF_\varphi : c > 0, \varphi : N \to \mathbb{R} \text{ a homomorphism}\},$$
where $F_\varphi \circ D = e^{\varphi(D)} F_\varphi$ for all $D \in N$ ([LP],[CG], see also [LS2]). Consequently the ergodic invariant Radon measures of the horocycle flow are
$$\{cm_\varphi : c > 0, \varphi : N \to \mathbb{R} \text{ a homomorphism}\},$$
where $m_\varphi \circ D = e^{\varphi(D)} m_\varphi$ for all $D \in N$.

There are periodic surfaces of exponential growth for which there are locally finite ergodic invariant measures which are not quasi–invariant with respect to some deck transformations, see [LS2].

Question 1. Does there exist an example of a (necessarily non–tame) surface $\Gamma \setminus \mathbb{D}$ with an h–ergodic invariant Radon measure which is not carried by a single orbit, and is not quasi invariant under the geodesic flow?

Question 2. What can be said about the infinite locally finite ergodic invariant measures for general unipotent flows on a homogenous space $\Gamma \setminus G$ when Γ is not a lattice? (The finite invariant measures are known [Rat1].)

2.4. Generic points.

At present, the generic points for horocycle flows on surfaces of infinite genus are only understood in the case of $\mathbb{Z}^d$–covers of compact surfaces.

Suppose M covers a compact surface M_0 in such a way that the group of deck transformations can be put in the form $\{D_{\underline{\xi}} : \underline{\xi} \in \mathbb{Z}^d\}$, where $D_{\underline{\xi}+\underline{\eta}} = D_{\underline{\xi}} \circ D_{\underline{\eta}}$. Choose some connected fundamental domain $\widetilde{M_0}$ for the action of the group of deck transformations on T^1M. Define the $\mathbb{Z}^d$*–coordinate* of $\omega \in T^1M$ to be the unique $\xi(\omega) \in \mathbb{Z}^d$ such that $\omega \in D_{\xi(\omega)}[\widetilde{M_0}]$ (Fig. 1c).

There is an analogy between the paths of the geodesic flow and the paths of a random walk on $\mathbb{Z}^d$. Define the *asymptotic drift of a vector* $\omega \in T^1M$ to be the following limit, if it exists:
$$\Xi(\omega) := \lim_{T\to\infty} \frac{1}{T}\xi(g^T\omega), \text{ where } g \text{ is the geodesic flow.}$$
Since g moves at unit speed, $\|\Xi(\omega)\|$ is uniformly bounded. Let
$$\mathfrak{C} := \text{closed convex hull of } \{\Xi(\omega) : \omega \in T^1M \text{ s.t. } \Xi(\omega) \text{ exists}\} \subset \mathbb{R}^d.$$

In the previous section we parametrized the ergodic invariant Radon measures of h by the way they transform under the deck transformations. One can also parametrize them by the almost sure value of $\Xi(\cdot)$:

Theorem 2.10. *Let M be a $\mathbb{Z}^d$–cover of a compact hyperbolic surface.*

1. *For every $\Xi \in int(\mathfrak{C})$ there exists an h–ergodic invariant Radon measure m_Ξ such that $\Xi(\cdot) = \Xi$ m_ξ–a.e., and this measure is unique up to scaling.*
2. *The volume measure m_0 is proportional to $m_{\underline{0}}$.*
3. *Every h–ergodic invariant Radon measure is proportional to m_Ξ for some $\Xi \in \mathrm{int}(\mathfrak{C})$.*

See [BL], and theorem 2.8.

Theorem 2.11 (S. & Schapira). *A vector $\omega \in T^1M$ is generic for some horocycle ergodic invariant Radon measure m iff $\Xi(\omega)$ exists and $\Xi(\omega) \in int(\mathfrak{C})$. In this case $m = cm_{\Xi(\omega)}$ for some $c > 0$. In particular, ω is generic for m_0 iff $\Xi(\omega) = 0$.*

Using the hyperbolicity of the geodesic flow and a standard specification argument, it is easy to construct vectors ω for which the limit $\Xi(\omega)$ does not exist. It is not difficult to arrange for ω to have a dense (horocycle) forward orbit. Thus there are abundantly many non-exceptional $\omega \in T^1M$ which are not generic for any Radon measure. This is yet another way in which h fails to be u.e. in the broad sense.

Question 1. What are the generic points for horocycle flows on nilpotent covers of compact hyperbolic surfaces?

Question 2. Suppose $M = \Gamma \setminus \mathbb{D}$ is Liouville (cf. theorem 2.1). Is it true that $\omega \in T^1M$ is generic for m_0 whenever $\frac{1}{T}\log F(\text{base point of } g^s(\omega)) \xrightarrow[T\to\infty]{} 0$ for all positive minimal eigenfunctions F? This is the case for compact surfaces, surfaces of finite area, and $\mathbb{Z}^d$–covers of compact surfaces.

2.5. Conditional unique ergodicity.

We continue to consider the special case of $\mathbb{Z}^d$–covers of compact hyperbolic surfaces.

We saw that there are infinitely many ergodic invariant measures. It turns out that up to scaling, only one of them – the volume measure – is non pathological from the ergodic theoretic point of view, in the sense that it admits a *generalized law of large numbers* in the sense of Aaronson [A2].

We explain what this means. Suppose φ is an ergodic measure preserving flow on a non–atomic measure space $(\Omega, \mathscr{B}, \mu)$, and fix some measurable set E of finite measure. We think of t as of "time" and of E as of an "event". The times when E "happened" are encoded by the function

$$x_{E,\omega}(t) := 1_E(\varphi^t(\omega)) = \begin{cases} 1 & \varphi^t(\omega) \in E; \\ 0 & \varphi^t(\omega) \notin E. \end{cases}$$

A generalized law of large numbers is a procedure for reconstructing $\mu(E)$ from $x_{E,\omega} : [0,\infty) \to \{0,1\}$:

Definition 2.2 (Aaronson). *A generalized law of large numbers (GLLN) is a function $L : \{0,1\}^{\mathbb{R}^+} \to [0,\infty)$, $L = L[x(\cdot)]$, such that for every $E \in \mathscr{B}$ of finite measure, $L[x_{E,\omega}(\cdot)] = \mu(E)$ for μ–a.e. ω.*

For example, if the underlying measure space is a probability space, then the ergodic theorem says that the following function is a GLLN:

$$L[x(t)] := \begin{cases} \lim\limits_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt & \text{the integral and limit exist} \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious how to change L to make it work when $0 < \mu(\Omega) < \infty$. But if $\mu(\Omega) = \infty$, then it is not clear how to proceed, because the ergodic theorem says that in this case $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T 1_E(\varphi^t(\omega))dt = 0$ for every E of finite measure.

It is natural to ask whether it is possible to find $a(T) = o(T)$ so that for every $E \in \mathscr{B}$, $\lim\limits_{T\to\infty} \frac{1}{a(T)}\int_0^T 1_E(\varphi^t(\omega))dt = \mu(E)$. This is never possible [A1]:

Theorem 2.12 (Aaronson). *Let φ be an ergodic measure preserving flow on an infinite σ–finite non–atomic measure space $(\Omega, \mathscr{B}, \mu)$. Suppose $f \in L^1$, $f > 0$. There is no $a(T) > 0$ s.t. $\frac{1}{a(T)}\int_0^T f(\varphi^t(\omega))dt$ converges a.e. to a constant $c \neq 0, \infty$.*

It is still possibile that there exists $a(T) > 0$ s.t. $\frac{1}{a(T)}\int_0^T f(\varphi^t(\omega))dt$ oscillates without converging to zero or infinity. One can hope for a summability method which forces convergence to $\int f d\mu$. Such "second order ergodic theorems" are considered in [ADF]. Here is such a theorem [LS3]:

Theorem 2.13 (Ledrappier–S.). *There exists $a(T) > 0$ s.t. for all $f \in L^1(m_0)$*

$$\lim_{N\to\infty} \frac{1}{\ln\ln N}\int_3^N \frac{1}{T\ln T}\left(\frac{1}{a(T)}\int_0^T f\circ h^s ds\right)dT = \int f dm_0 \quad m_0\text{-a.e.}$$

The corresponding GLLN is $L[x(t)] := \lim\limits_{N\to\infty} \frac{1}{\ln\ln N}\int_3^N \frac{1}{T\ln T}\left(\frac{1}{a(T)}\int_0^T x(s)ds\right)dT$ when the limit make sense, and $L[x(t)] := 0$ otherwise.

We decribe $a(T)$. Recall the definitions of $\widetilde{M}_0$ and of the $\mathbb{Z}^d$–coordinate ξ from §9. We pick $\omega \in \widetilde{M}_0$ randomly according to the uniform distribution on $\widetilde{M}_0$, $m_0(\,\cdot\,|\widetilde{M}_0)$, and consider the random variables $\omega \mapsto \xi(g^T(\omega))$. It follows from the work of Ratner [Rat2] and Katsuda & Sunada [KS] that $\xi(g^T(\omega))/\sqrt{T}$ converges in distribution to a non–degenerate multivariate Gaussian random variable N on $\mathbb{R}^d$. If $\mathrm{Cov}(N)$ is the covariance matrix of N, and $\sigma := \sqrt[d]{|\det \mathrm{Cov}(N)|}$, then

$$a(T) = \frac{m_0(\widetilde{M}_0)}{(4\pi\sigma)^{d/2}}\frac{T}{(\ln T)^{d/2}}.$$

Theorem 2.13 also holds for $\mathbb{Z}^d$–covers of non–compact surfaces of finite area, but with different $a(T)$ [LS1].

There are no similar results for any of the other h–ergodic invariant Radon measures. The reason is tied to the following property:

Definition 2.3 (Aaronson). *An ergodic invariant measure m for a flow φ (or a map T) is called* squashable*, if there is a measurable map Q which commutes with φ (or T) such that $m \circ Q^{-1} = cm$ with $c \neq 0, 1$.*

Squashable measures do not admit GLLN's: Suppose there were a GLLN $L[\cdot]$. Choose a measurable set E of positive finite measure, and some ω s.t. $L[1_A(h^t v)] = m(A)$ for $A = E, Q^{-1}E$ and $v = \omega, Q(\omega)$. We have

$$m(E) = L[1_E(h^s Q\omega)] = L[1_E(Qh^s\omega)] = L[1_{Q^{-1}E}(h^s\omega)] = m(Q^{-1}E) = cm(E),$$

whence $c = 1$, a contradiction. Thus no GLLN can exist.

Any locally finite h–ergodic invariant Radon measure m which is not proportional to m_0 is squashable, because by theorem 2.8 such a measure satisfies $m \circ D_{\underline{\xi}} = e^{\langle \underline{a}, \underline{\xi} \rangle} m$ for some vector $\underline{a} \neq \underline{0}$ and all deck transformations $D_{\underline{\xi}}$, and all deck transformations commute with h, being isometries. As a result we obtain the following "conditional unique ergodicity" result [LS2]:

Theorem 2.14 (Ledrappier – S.). *The horocycle flow on a $\mathbb{Z}^d$–cover of a compact hyperbolic surface has, up to scaling, exactly one ergodic invariant Radon measure which admits a GLLN: the volume measure m_0.*

3. Non-Compact Group Extensions of Uniquely Ergodic Transformations

3.1. Group extensions. Suppose $T : \Omega \to \Omega$ is a bimeasurable bijection on a standard measurable space $(\Omega, \mathscr{B})$. Let G be a locally compact second countable topological group with left Haar measure m_G, with $m_G(G) = 1$ when G is compact. Fix a Borel function $\varphi : \Omega \to G$.

Definition 3.1. *The* skew–product *with* base $T : \Omega \to \Omega$, *and* cocycle $\varphi : \Omega \to G$ *is the map $T_\varphi : \Omega \times G \to \Omega \times G$ given by $T_\varphi : (\omega, g) \mapsto \big(T(\omega), g\varphi(\omega)\big)$. Such maps are called* group extensions*.*

In the cases considered below, T is a homeomorphism of a topological space Ω which is either a compact metric space, or a compact metric space Ω_0 minus a countable collection of points. With such examples in mind, we call a measure m on $\Omega \times G$ *locally finite*, if $m(\Omega \times K) < \infty$ for all compact $K \subset \Omega$.

If μ is a T–invariant probability measure, then $m_0 := \mu \times m_G$ is a locally finite T_φ–invariant measure, although it is not always ergodic (e.g. when φ can be put in the form $\varphi = u(u \circ T)^{-1}$ with u Borel). The basic measure rigidity result for compact group extensions is [P], [F2]:

Theorem 3.1 (Furstenberg – Parry). *Let T be a uniquely ergodic homeomorphism of a compact metric space Ω, G be a compact Abelian group, and $\varphi : \Omega \to G$ be continuous. T_φ is uniquely ergodic iff m_0 is T_φ–ergodic.*

If G is not compact, then there could be other measures: let $\alpha : G \to \mathbb{R}$ be a measurable homomorphism, and suppose there is a probability measure ν_α on Ω s.t. $\frac{d\nu_\alpha \circ T}{d\nu_\alpha} = \exp[\alpha \circ \varphi]$, then the measure

$$dm_\alpha(\omega, g) = e^{-\alpha(g)} d\nu_\alpha(\omega) dm_G(g) \tag{3.1}$$

is a locally finite invariant measure for T_φ, as can be verified by direct calculation. Such measures are called *Maharam measures*. Some remarks:

1. If $G = \mathbb{R}$ and $\alpha = id$, then $\varphi = \log \frac{d\nu \circ T}{d\nu}$ and T_φ is called the *Radon-Nikodym extension* of T. T_φ preserves m_α, even when T does not preserve ν. This was Maharam's original motivation [M].
2. Suppose $\alpha \equiv 0$, then ν_0 is T–invariant and $m_0 = \nu_0 \times m_G$. If G is compact, then this is the only possibility, because all measurable homomorphisms $\alpha : G \to \mathbb{R}$ are trivial.
3. Maharam measures m_α with $\alpha \not\equiv 0$ do not admit GLLN's, because they are squashable: if $Q_h : (\omega, g) \mapsto (\omega, hg)$ and $h \notin \ker \alpha$, then $Q_h \circ T_\varphi = T_\varphi \circ Q_h$ and $m_\alpha \circ Q_h = c m_\alpha$ where $c = e^{-\alpha(h)} \neq 1$.

There is an obvious generalization of Maharam's construction to skew–produts over group actions. Burger's measure (2.4) and the measures arising from Babillot's bijection (2.5) are Maharam measures for the skew–product action (2.3).

The following questions arise naturally [ANSS]: Given a u.e. T, a cocycle $\varphi : \Omega \to G$, and a measurable homomorphism $\alpha : G \to \mathbb{R}$, does the Maharam measure m_α exist, and is it unique? Is it ergodic? Is every locally finite ergodic invariant measure proportional to a Maharam measure?

The following statement comes close to saying that every locally finite ergodic invariant measure is "Maharam like", after suitable change of coordinates [Rau].

Theorem 3.2 (Raugi). *If m is a locally finite T_φ–ergodic invariant measure on $\Omega \times G$, then there are a closed subgroup $H \subset G$ and Borel function $u : \Omega \to G$ s.t.*

1. *if $\widetilde{\varphi}(x) := u(x)\varphi(x)u(Tx)^{-1}$, then $\widetilde{\varphi}(x) \in H$ for m a.e. $(x, g) \in \Omega \times G$;*
2. *if $\vartheta : (x, g) \mapsto (x, gu(x)^{-1})$, then $m \circ \vartheta^{-1}$ is a $T_{\widetilde{\varphi}}$–ergodic invariant measure supported on $\Omega \times H$, and there exists a measurable homomorphism*

$\alpha : H \to \mathbb{R}$ and a σ–finite measure ν_α on Ω s.t. $\frac{d\nu_\alpha \circ T}{d\nu_\alpha} = \exp[\alpha \circ \widetilde{\varphi}]$ and

$$dm \circ \vartheta^{-1}(\omega, h) = e^{-\alpha(h)} d\nu_\alpha(\omega) dm_H(h). \tag{3.2}$$

3. But in general ν_α and $m \circ \vartheta^{-1}$ need not be locally finite.

The case $G = \mathbb{R}^n \times \mathbb{Z}^m$ was done in [Sa2].

The significance of part (3) is that there is an abundance of infinite σ–finite solutions to the equation $d\nu_\alpha \circ T / d\nu_\alpha = \exp[\alpha \circ \widetilde{\varphi}]$. The challenge is to determine which of them has the property that $m = (e^{-\alpha(h)} d\nu_\alpha(\omega) dm_H(h)) \circ \vartheta$ is locally finite. In some cases, and using additional structure, one can show that $H = G$ or that u is essentially bounded (i.e. $u(\omega) \in K$ a.e. for some K compact). In such cases m is locally finite iff ν_α is finite. We discuss two examples below.

3.2. Cylinder Transformations. The first example we consider is a group extension of the irrational rotation $T_\theta : \mathbb{T} \to \mathbb{T}$, $T_\theta : \omega \mapsto (\omega + \theta) \bmod 1$, where θ is a fixed irrational number and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The cocycle is

$$\varphi : \mathbb{T} \to \mathbb{Z}\ ,\ \varphi(\omega) := \begin{cases} 1 & 0 \le \omega < \frac{1}{2} \\ -1 & \frac{1}{2} \le \omega < 1. \end{cases}$$

Let $T_{\theta,\varphi} := (T_\theta)_\varphi$, then $T_{\theta,\varphi} : (\omega, n) \mapsto ((\theta + \alpha) \bmod 1, n + \varphi(\omega))$. Note that φ and $T_{\theta,\varphi}$ are not continuous.

The original motivation was the theory of random walks [AK]. The iterates of a G–extension T_φ are given in general by

$$T_\varphi^n(\omega, g) = \left(T^n(\omega), g\varphi(\omega)\varphi(T\omega)\cdots\varphi(T^{n-1}\omega)\right). \tag{3.3}$$

The second coordinate is a random walk on G started at g. The function φ controls the jumps, and the map $T : \Omega \to \Omega$ is the driving noise. For example, if $\Omega = \{0,1\}^{\mathbb{N}}$, T is the left shift $(T\omega)_i = \omega_{i+1}$ together with the Bernoulli $(\frac{1}{2}, \frac{1}{2})$–measure, and $\varphi : \Omega \to \mathbb{Z}$ is the function $\varphi(\omega) = (-1)^{\omega_0}$, then the second coordinate in (3.3) is the simple random walk on $\mathbb{Z}$ (started at g). The interest in the cylinder transformation $T_{\theta,\varphi}$ is that the random walk it generates is driven by a map with entropy zero. Another reason T_φ is interesting is that it appears as the Poincaré section for the linear flow on the staircase surface, see below.

The measure $m_0 := m_{\mathbb{T}} \times m_{\mathbb{Z}}$ (Lebesgue times counting measure) is an invariant Radon measure for $T_{\theta,\varphi}$. This measure is ergodic [CK], see also [Sch1]. Nakada showed that Maharam's construction yields additional locally finite ergodic invariant measures [N1], [N2]:

Theorem 3.3 (Nakada). *For every $\theta \notin \mathbb{Q}$ and $\alpha \in \mathbb{R}$ there is a unique probability measure ν s.t. $\frac{d\nu \circ T_\theta}{d\nu} = \exp(\alpha\varphi)$. The measure $dm_\alpha(\omega, n) := e^{-\alpha\varphi(\omega)} d\nu(\omega) dm_{\mathbb{Z}}(n)$ is a conservative ergodic invariant Radon measure for $T_{\theta,\varphi}$.*

Theorem 3.4 (Aaronson, Nakada, S., & Solomyak). *Suppose $\theta \notin \mathbb{Q}$, then every ergodic invariant Radon measure for $T_{\theta,\varphi}$ is proportional to m_α for some $\alpha \in \mathbb{R}$.*

For more complicated step functions φ, see [ANSS] and [C].

If $\alpha \neq 0$, then m_α is squashable, and therefore does not admit a GLLN. Aaronson & Keane have shown in [AK] that m_0 is not squashable. In fact it admits a GLLN. This is a particular case of the following general result [A1]:

Theorem 3.5 (Aaronson). *Let $T : \Omega \to \Omega$ be a translation on a compact metric group Ω, and suppose $\varphi : \Omega \to \mathbb{Z}^d$ is measurable. Let m_0 be the product of the Haar measures on Ω and $\mathbb{Z}^d$. If m_0 is ergodic, then m_0 admits a GLLN.*

Corollary 3.1. *Suppose θ is irrational, then up to scaling, $T_{\theta,\varphi}$ has exactly one ergodic invariant Radon measure with a GLLN: m_0.*

The GLLN presented in theorem 2.13 is *finitely observable* in the sense that the knowledge of $\{1_E(\varphi^t\omega)\}_{0\le t\le T}$ for finite T yields an approximation to $m(E)$ which tends a.s. to $m(E)$ as $T \to \infty$. The GLLN provided by the existing proof of theorem 3.5 does not seem to be finitely observable.

If we assume more on θ, then we can exhibit a finitely observable GLLN, using the theory of *rational ergodicity* [A1], [A3].

Definition 3.2 (Aaronson). *A conservative ergodic measure preserving map τ on a σ–finite measure space $(\Omega, \mathscr{B}, m)$ is called* rationally ergodic*, if there are $M > 0$ and a set $A \in \mathscr{B}$ with finite positive measure s.t. for all $n \geq 1$,*

$$\left[\int_A \left(\sum_{k=0}^{n-1} 1_A \circ \tau^k\right)^2 dm\right]^{1/2} \leq M \left[\int_A \left(\sum_{k=0}^{n-1} 1_A \circ \tau^k\right) dm\right]. \qquad (3.4)$$

(The other direction to Cauchy–Schwarz.)

Rationally ergodic maps admit GLLN's. To describe them, we use the following notation for Cesàro convergence: $\underset{k\to\infty}{\mathrm{CLim}}\, x_k := \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N x_k$.

Theorem 3.6 (Aaronson). *Let τ be a rationally ergodic map on the space $(\Omega, \mathscr{B}, m)$, fix some A of finite positive measure which satisfies* (3.4)*, and set*

$$a_n := \frac{1}{m(A)^2} \int_A \sum_{k=1}^{n-1} 1_A \circ \tau^k dm.$$

There are $n_k \uparrow \infty$ s.t. for every $f \in L^1$, $\underset{k\to\infty}{\mathrm{CLim}} \left[\frac{1}{a_{n_k}} \sum_{j=0}^{n_k-1} f\circ\tau^j\right] = \int f dm$ a.e.

The sequence a_n is called the *return sequence of* τ. It is unique up to asymptotic equivalence, see [A1].

Aaronson & Keane proved in [AK] that if θ is an irrational quadratic surd, then m_0 is rationally ergodic with return sequence $a_n \asymp n/\sqrt{\log n}$ ($a_n \asymp b_n$ means $C^{-1} \leq a_n/b_n \leq C$ for some $C > 0$ and all n large enough). It follows that

Theorem 3.7. *Suppose θ is an irrational root of a quadratic polynomial with integer coefficients, then $T_{\theta,\varphi}$ has, up to scaling, a unique ergodic invariant Radon measure with a GLLN: m_0. This GLLN takes the form*

$$L[x(n)] := \begin{cases} \underset{k\to\infty}{\text{CLim}} \left[\frac{1}{a_{n_k}} \sum_{j=0}^{n_k-1} x(j)\right] & \textit{the limit exists} \\ 0 & \textit{otherwise} \end{cases}$$

for some sequences $n_k \uparrow \infty$ and $a_n \asymp n/\sqrt{\log n}$.

Question 1. What are the generic points for m_0?

We finish our discussion of T_φ with the following nice construction due to Hubert and Weiss [HW]. Let $\{R_k\}_{k\in\mathbb{Z}}$ be the sequence of tagged rectangles $R_k := [0,2] \times [0,1] \times \{k\}$ minus the points with integer coordinates. We denote the left and right vertical sides of R_k by l_k and r_k, and the top and bottom horizontal sides by t_k, b_k. For each k,

- glue l_k to r_k by the map $(x, y; k) \mapsto (x + 2, y; k)$;
- glue the left half of t_k to the right half of b_{k-1} by the map $(x, y; k) \mapsto (x + 1, y - 1; k - 1)$;
- glue the right half of t_k to the left half of b_{k+1} by the map $(x, y; k) \mapsto (x - 1, y - 1; k + 1)$.

The result is a surface of infinite area and infinite genus, which we denote by M. Fix an angle β, and let $\varphi_\beta : M \to M$ be the flow which moves each point x at unit speed on the line with slope $\tan\beta$ passing through x, while respecting identifications (Fig. 2).

Theorem 3.8 (Hubert & Weiss). *Suppose* $\tan\beta$ *is irrational, and let* $Q : M \to M$ *be the map* $Q(x, y; k) = (x, y; k + 1)$*, then*

1. *For every $\alpha \in \mathbb{R}$ there exists up to scaling exactly one ergodic invariant Radon measure m_α such that $m_\alpha \circ Q = e^\alpha m_\alpha$;*
2. *All ergodic invariant Radon measures are of this form.*

The proof is done by first checking that the union of the horizontal sides of R_k forms a Poincaré section with the properties that the roof function is constant, and the Poincaré map is conjugate to some $T_{\theta,\varphi}$ with $\theta = \theta(\beta)$ irrational [HW].

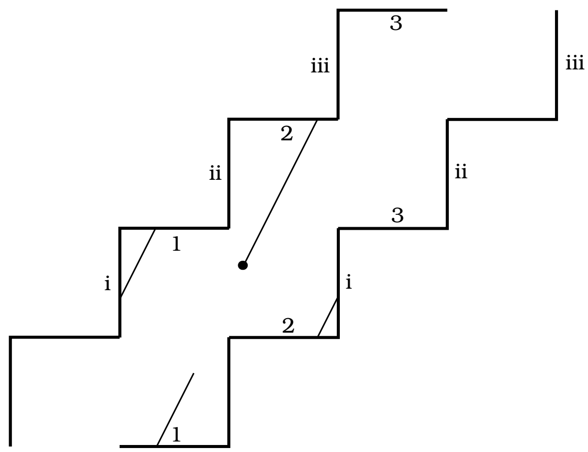

Figure 2. The linear flow on the staircase surface

Imagining other translation surfaces, one is led to the following question:

Question 2. What can be said about the locally finite ergodic invariant measures for skew products over "typical" interval exchange transformations and step function cocycles?

P. Hooper has recently obtained some very interesting related results [Hoo].

3.3. Hajian-Ito-Kakutani Maps.

This example comes from the world of symbolic dynamics. Recall that the horocycle flow parametrizes the strong stable foliation of the geodesic flow: $\{h^t(\omega)\}_{t\in\mathbb{R}} = \{\omega' \in T^1M : d(g^s\omega', g^s\omega) \xrightarrow[s\to\infty]{} 0\}$. The HIK map parametrizes the symbolic dynamical analogue of the stable foliation (tail relation) for a skew–product over a subshift of finite type.

Let $\sigma : \Sigma_A^+ \to \Sigma_A^+$ be a one–sided *subshift of finite type*. This means that there is a finite set $S = \{0, \ldots, N\}$ and a matrix of zeroes and ones $A = (t_{ab})_{S\times S}$ so that

$$\Sigma_A^+ := \{(x_0, x_1, \ldots) \in S^{\mathbb{N}} : \forall i \geq 0,\ t_{x_i x_{i+1}} = 1\},$$
$$\text{and } \sigma : (x_0, x_1, \ldots) \mapsto (x_1, x_2, \ldots).$$

Endow Σ_A^+ with the metric $d(x, y) := \exp[-\min\{n \geq 0 : x_n \neq y_n\}]$. This map is *expansive*: if $d(\sigma^n x, \sigma^n y) < 1$ for all n, then $x = y$. It is topologically mixing iff there is an m s.t. all the entries of A^m are positive.

Fix some continuous function $f : \Sigma_A^+ \to \mathbb{R}^d$. The system playing the role of the geodesic flow is the (discrete time) map $\sigma_f : \Sigma_A^+ \times \mathbb{R}^d \to \Sigma_A^+ \times \mathbb{R}^d$

$$\sigma_f : (x, \xi) \mapsto (\sigma(x), \xi + f(x)).$$

We metrize $\Sigma_A^+ \times \mathbb{R}^d$ by $d((x,\xi),(y,\eta)) := d(x,y) + \|\xi - \eta\|$. One can check, using the expansivity of σ, that $d(\sigma_f^n(x,\xi), \sigma_f^n(y,\eta)) \xrightarrow[n\to\infty]{} 0$ iff

$$\exists n \text{ s.t. } \sigma^n(x) = \sigma^n(y) \text{ and } \xi - \eta = \sum_{k=0}^{\infty}[f(\sigma^k y) - f(\sigma^k x)]. \qquad (*)$$

(The sum always converges, in fact all terms with $k \geq n$ vanish.) If $(x,\xi),(y,\eta)$ satisfy $(*)$, then we write $(x,\xi) \overset{f}{\sim} (y,\eta)$. This is an equivalence relation. For an example how this equivalence relation appears as the symbolic dynamical coding of "real" foliations, see [BM] and [PoS].

Our task is to construct a map whose orbits are the equivalence classes of $\overset{f}{\sim}$. Such a map can be easily constructed using Vershik's adic transformations [V]. Here is the construction. Define $\preceq$ to be the *reverse lexicographic order* on Σ_A^+:

$$x \preceq y \Leftrightarrow \exists n \text{ s.t. } \big(x_n \leq y_n \text{ and } x_{n+k} = y_{n+k} \text{ for all } k \geq 1\big).$$

Two points x, y are $\preceq$–comparable iff $\exists n$ s.t. $\sigma^n(x) = \sigma^n(y)$. In this case we write $x \sim y$. If $x \sim y$, then there are only finitely many points between x and y (at most $|S|^n$). It follows that for all x not equivalent to a $\preceq$–maximal or minimal point, the set $\{y : y \sim x\}$ has the same order structure as $\mathbb{Z}$.

One can check that x is equivalent to a maximal (resp. minimal) point iff $\sigma^n(x)$ is maximal (resp. minimal) for some n. This leads to the following definition:

Definition 3.3 (Vershik). *Let* $\Omega := \Sigma_A^+ \setminus \{x : \exists n\, s.t.\, \sigma^n(x)$ *is maximal or minimal*$\}$. *The* adic transformation *of* Σ_A^+ *is the map* $T : \Omega \to \Omega$, $T(x) := \min\{y \in \Omega : y \succneqq x\}$.

The point is that for every $x \in \Omega$, $\{T^n(x) : n \in \mathbb{Z}\} = \{y : y \sim x\}$ for $x \in \Omega$. To get a map whose orbits are the equivalence classes of $\overset{f}{\sim}$, we make the following definition.

Definition 3.4 (Hajian–Ito–Kakutani). *Let* $f : \Sigma_A^+ \to \mathbb{R}^d$ *be a continuous function. The* HIK cocycle for f *is*

$$\varphi(x) := \sum_{k=0}^{\infty}[f(\sigma^k x) - f(\sigma^k T x)].$$

The HIK map *is* $T_\varphi : \Omega \times \mathbb{R}^d \to \Omega \times \mathbb{R}^d$, $T_\varphi : (x,\xi) \mapsto (T(x), \xi + \varphi(x))$.

A direct calculation shows that $\{T_\varphi^n(x,\xi)\}_{n\in\mathbb{Z}} = \{(y,\eta) : (y,\eta) \overset{f}{\sim} (x,\xi)\}$, and so $\{T_\varphi^n(x,\xi)\}_{n\in\mathbb{Z}} = \{(y,\eta) : d(\sigma_f^n(x,\xi), \sigma_f^n(y,\eta)) \xrightarrow[n\to\infty]{} 0\}$.

Here is an example [HIK],[AW]. Suppose $\Sigma_A^+ = S^{\mathbb{N}}$. The unique maximal point is $(N,N,N,\dots)$, the unique minimal point is $(0,0,0,\dots)$, and T is the map

$$T : (\underbrace{N,\dots,N}_{n},k,*) \mapsto (\underbrace{0,\dots,0}_{n},k+1,*) \quad (k<N, n\geq 0) \tag{3.5}$$

Informally, T "adds one with carry to the right". Formula (3.5) makes sense for all points in $S^{\mathbb{N}} \setminus \{(1,1,1,\dots)\}$. If we define $T(1,1,1,\dots) := (0,0,0,\dots)$, then we obtain a homeomorphism of $S^{\mathbb{N}}$, widely known under the name the *adding machine*.

Now fix some probability vector $\underline{p}_0 := (p_0,\dots,p_N)$ on S all of whose coordinates are non–zero, let $f : \Sigma_A^+ \to \mathbb{R}$ denote the function $f(x) = -\log p_{x_0}$, and define φ to be the HIK cocycle of f. A direct calculation shows that

$$\varphi = \log\left(\frac{d\nu_0\circ T}{d\nu_0}\right),$$

where ν_0 is the Bernoulli measure of $\underline{p}_0$. This measure is in general *not* T–invariant. But the measure $e^{-t}d\nu_0(\omega)dt$ is T_φ–invariant. Similarly, given $\alpha\in\mathbb{R}$, let $\underline{p}_\alpha$ denote the probability vector proportional to $(p_0^\alpha,\dots,p_N^\alpha)$, and let ν_α denote the corresponding Bernoulli measure on $\Sigma_A^+ = S^{\mathbb{N}}$. Then $\log\left(\frac{d\nu_\alpha\circ T}{d\nu_\alpha}\right) = \alpha\varphi$, so $m_\alpha := e^{-\alpha t}d\nu_\alpha(\omega)dt$ is T_φ–invariant for every $\alpha\in\mathbb{R}$.

These measures are not always ergodic. The simplest example of this is when $\underline{p} = (b,\dots,b)$ where $b = 1/|S|$. In this case φ takes values in $b\mathbb{Z}$, and the function $F(\omega,\xi) = \exp[2\pi i\xi/b]$ is T_φ–invariant. The ergodic components of m_α take the form $e^{-\alpha t}d\nu_\alpha(\omega)dm_{b\mathbb{Z}+c}$ where $m_{b\mathbb{Z}+c}$ is the counting measure on the coset $b\mathbb{Z}+c$, and $0\leq c<b$ [AW],[HIK]. We call this phenomenon the *lattice phenomenon*.

We now turn to the case of general HIK maps, assuming only that $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is topologically mixing, and that $f : \Sigma_A^+ \to \mathbb{R}^d$ has *summable variations*:

$$\sum_{n=1}^{\infty} \mathrm{var}_n f < \infty, \text{ where } \mathrm{var}_n f := \sup\{f(x)-f(y) : x_i = y_i\ (i=0,\dots,n-1)\}.$$

We denote $f_n := f + f\circ\sigma + \cdots + f\circ\sigma^{n-1}$. Let H_f denote the smallest closed subgroup of $\mathbb{R}^d$ which contains $\{f_n(x) - f_n(y) : \sigma^n(x) = x, \sigma^n(y) = y, n\in\mathbb{N}\}$. The following fact can be found in [Sa2] (see also [PaS])

Lemma 3.1. *There exists a function $u_f : \Sigma_A^+ \to \mathbb{R}^d$ with summable variations and a constant c_f such that $\widetilde{f} := f + u_f - u_f\circ\sigma + c_f$ takes values in H_f.*

The group H_f is invariant under addition of coboundaries and constants, so one cannot reduce the range of f further by means of a continuous coboundary.

Let φ and $\widetilde{\varphi}$ be the HIK cocycles of f and $\widetilde{f}$, respectively. Direct calculations show that $\widetilde{\varphi} = \varphi + u_f - u_f \circ T$ and that the image of $\widetilde{\varphi}$ is in H_f. The map $\vartheta(\omega, \xi) = (\omega, \xi + u_f(\omega))$ satisfies $\vartheta^{-1} \circ T_\varphi \circ \vartheta = T_{\widetilde{\varphi}}$. We see that if $H_f \neq \mathbb{R}^d$, then T_φ is conjugate to an HIK map exhibiting the lattice phenomenon.

We describe the invariant measures of T_φ.

Theorem 3.9.

1. *For every $\alpha \in \mathbb{R}^d$ there is a unique probability measure ν_α s.t. $\frac{d\nu_\alpha \circ T}{d\nu_\alpha} = e^{\langle \alpha, \varphi \rangle}$;*

2. *If $H_f = \mathbb{R}^d$, then $m_\alpha := e^{-\langle \alpha, t\rangle} d\nu_\alpha(\omega) dt$ is a T_φ–ergodic invariant locally finite measure;*

3. *If $H_f = \mathbb{R}^d$, then every T_φ–ergodic invariant locally finite measure is proportional to m_α for some $\alpha \in \mathbb{R}^d$.*

Part 1 is in [PeS], see also [ANSS]. Part 2 is because σ_f is m_α–exact [G] (see [ANSS] for details). Part 3 was proved under the assumption that f is locally constant in [ANSS] and in the general case in [Sa2].

Next we discuss the lattice case. For every $c \in \mathbb{R}^d / H_f$, let $m_{H_f + c}$ denote the measure on the coset $H_f + c$ induced by the Haar measure on H_f.

Theorem 3.10. *Suppose $H_f \neq \mathbb{R}^d$, let $\widetilde{f} := f + u_f - u_f \circ \sigma + c_f$ where u_f, c_f are given by lemma* 3.1*, and let $\widetilde{\varphi}$ denote the HIK cocycle of $\widetilde{f}$.*

1. *The locally finite ergodic invariant measures for $T_{\widetilde{\varphi}}$ are the measures proportional to $m_{\alpha,c} := e^{-\langle \alpha, t \rangle} d\nu_\alpha(\omega) dm_{H_f + c}(t)$ for some $\alpha \in \mathbb{R}^d$ and $c \in \mathbb{R}^d / H_f$.*

2. *The locally finite ergodic invariant measures for T_φ are the measures proportional to $m_{\alpha,c} \circ \vartheta$ ($\alpha \in \mathbb{R}^d, c \in \mathbb{R}^d / H_f$), where $\vartheta : (\omega, \xi) \mapsto (\omega, \xi + u_f(\omega))$.*

Theorem 3.10 was proved for $f : \Sigma_A^+ \to \mathbb{Z}^d$ s.t. $H_f = \mathbb{Z}^d$ in [ANSS], and in the general case in [Sa2].

These results show that the group H mentioned in theorem 3.2 is always equal to H_f, and that the measurable function u there can be chosen to be bounded (in fact with summable variations). Consequently the change of coordinates ϑ preserves local finiteness, and the problems mentioned in part (3) of that theorem do not arise. For examples of skew products where these problems do arise, see [Sa2],[Rau].

Finally we consider the problem of GLLN's. Here we need the stronger assumption that f is Hölder continuous. Under this assumption it is proved in [ANSS] that m_0 is rationally ergodic (cf. definition 3.2). Since all other measures are squashable, we obtain

Theorem 3.11. *If $H_f = \mathbb{R}^d$, then T_φ has, up to scaling a unique locally finite ergodic invariant measure with a GLLN: m_0. The GLLN takes the form*

$$L[x(n)] := \begin{cases} \operatorname*{CLim}_{k\to\infty} \left[\frac{1}{a_{n_k}} \sum_{j=0}^{n_k-1} x(j)\right] & \text{the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

for some sequences $n_k \uparrow \infty$ and $a_n \asymp n/(\log n)^{d/2}$.

For an interesting application to the study of the stable foliation for a pseudo–Anosov diffeomorphism, see [PoS].

The generic points of certain HIK maps can be described. This is ongoing work with J. Aaronson, and will be published elsewhere.

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