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RESEARCH ARTICLE

Modulo Sampling of FRI Signals

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ABSTRACT An analog-to-digital converter (ADC) 's dynamic range is critical during sampling analog signals. A modulo operation prior to sampling can be used to enhance the effective dynamic range of the ADC. Further, the sampling rate of ADC also plays a crucial role, and it is desirable to reduce it. The finite-rate-of-innovation (FRI) signal model, ubiquitous in many applications, can be used to reduce the sampling rate. In the context of modulo folding for FRI sampling, existing works operate at a very high sampling rate compared to the rate of innovation (RoI) and require a large number of samples compared to the degrees of freedom (DoF) of the FRI signal. Moreover, these approaches use infinite-length filters that are practically infeasible. We consider the FRI sampling problem with a compactly supported kernel under the modulo framework. We derive theoretical guarantees and show that sampling above the RoI could uniquely identify FRI signals. The number of samples for identifiability is equal to the DoF. We propose a practical algorithm to estimate the FRI parameters from the modulo samples. We show that the proposed approach has the lowest error in estimating the FRI parameters while operating with the lowest number of samples and sampling rates compared to existing techniques. The results help design cost-effective, high-dynamic-range ADCs for FRI signals.

INDEX TERMS Finite-rate-of-innovation (FRI) signals, sub-Nyquist sampling, modulo sampling, high-dynamic range ADCs, unlimited sampling.

I. INTRODUCTION

Finite-rate-of-innovation (FRI) signals have few degrees of freedom, which aids in sub-Nyquist sampling [1]. A popular FRI signal model is one where the signal consists of a linear combination of delayed copies of a known pulse. Such a model is ubiquitous in several time-of-flight applications such as radar and ultrasound imaging [2], [3], [4], [5], [6], [7], [8]. A typical sampling and reconstruction framework for such signals consists of a tailor-made sampling kernel followed by a sampler or analog-to-digital converter (ADC) and a parameter estimation block. The sampling kernels are designed to spread the FRI signal information such that by using a low-rate, finite number of samples, the parameter-estimation block estimates the time-delays and amplitudes [7], [9], [10], [11], [12]. The key focus of prior works on FRI was to reduce the sampling rate of the ADCs and, consequently, their cost and power consumption.

Apart from the sampling rate, another critical parameter of an ADC is its dynamic range. The signal's dynamic range should be well within the ADC's dynamic range. Otherwise, the signal will be clipped, and perfect recovery is not guaranteed from the samples. In the FRI framework, the dynamic range of the signals may vary significantly due to a large variation in the amplitudes of the targets in time-of-flight imaging applications. Hence, the problem of designing an ADC that can sample a wide range of FRI signals without clipping and simultaneously operating at the lowest possible rate is of great importance. This problem is the focus of this paper.

A *modulo* preprocessing step can be used prior to sampling to address the dynamic range issue. The modulo operation folds or wraps the signal to keep it within the ADC's dynamic range prior to sampling. In this way, the ADC can sample signals with a very high dynamic range. Due to the

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non-linearity of the modulo operation, perfect reconstruction of the original signal from the modulo or wrapped signal samples may not be possible, and additional steps may be required.

Bhandari et al. [13] showed that theoretically, it is possible to reconstruct a bandlimited signal from its modulo samples perfectly, provided that the sampling rate is higher than the Nyquist rate. The authors also proposed an algorithm based on the higher-order differences (HoD) to unwrap the modulo samples up to a constant factor. In this algorithm, the oversampling factor (OF) is $(2\pi e) \approx 17$, where e is the Euler's constant. Romonov and Ordentlich [14] proposed an alternative method where the folded samples can be estimated from the unfolded ones by using a linear prediction (LP) approach provided that OF > 1. Bhandari et al. [15] presented results for periodic bandlimited (PBL) signals. They showed that the folding instants of the modulo signal could be determined by using the out-of-band Fourier coefficients of the modulo signal. The sampling rate is proportional to the number of Fourier coefficients to be determined, which depends on the number of folding instants. Although an upper bound on the number of folding instants is not provided, it increases as the dynamic range of the ADC decreases [15]. Recently, we proposed a robust and sampling-efficient unfolding algorithm for bandlimited signals [16], [17]. The algorithm is shown to operate close to the Nyquist rate compared to existing algorithms for bandlimited signals.

There are several extensions of modulo sampling for different signal models and tasks such as wavelets [18], a mixture of sinusoids [19], multi-dimensional signals [20], sparse vector recovery [21], [22], the direction of arrival estimation problem [23], [24], [25], computed tomography [26], graph signals [27], neural recording [28], spike covariance estimation [29], modulo hysteresis [30], [31], [32], denoising modulo samples [33], [34], [35], [36], bandpass signals [37], multiple-input multiple-output systems [38], array signal processing [39], and more. Extensions were also proposed for FRI signals [40], [41]. In [40] and [41], the authors used a periodic lowpass filter (LPF) as a sampling kernel, because of which the filtered signal can be written as a PBL signal. The signals are then folded and sampled. For reconstruction, in [40], the HoD-based approach [13] is first applied for unfolding, and then the annihilating filter (AF) method is used for FRI parameter estimation [42]. In [41], authors used a Fourier-Prony (FP) method proposed in [15] where the residual signal (difference between folded samples and the true/unfolded samples) is first estimated and then used AF to determine FRI parameters. This approach requires knowledge of the number of foldings of the signal. In [41], authors proposed to use a model-order estimation approach to determine the number of folds. Recently, a two-channel approach was proposed to sample FRI signals where the sampling can be done at the rate of innovation at each channel [43].

The unfolding results discussed for bandlimited signals [13], [14] and for PBL [15] can be extended to FRI signals by using an LPF and a periodic-LPF, respectively. For example, the LPF output of an FRI signal is bandlimited. Hence, unwrapping algorithms of [13] or [14] can be used in the unfolding step of the two-stage reconstruction process discussed above. However, due to the infinite support of the LPF and the filtered output, countably infinite unfolded samples are required to determine the Fourier samples for FRI recovery. Similarly, one can use a periodic LPF as in [40] or a sum-of-sincs (SoS) kernel [7], [12] to convert an FRI signal to an equivalent PBL signal. Then, the algorithm in [15] can be used for unfolding. However, theoretical identifiability results for PBL signals are not derived in [15]. The identifiability results for bandlimited signals from modulo-folded samples cannot be extended to the PBL signal model. This is because of the discrete nature of the spectrum in the periodic case as opposed to the continuous spectra of convention bandlimited signals.

From the above survey, we summarize the shortcomings of existing approaches:

- The sampling kernel is a critical component in an FRI sampling framework. Typically, it is desirable to have compactly supported sampling kernels from a practical implementation aspect. However, existing approaches for modulo recovery are based on infinite support sampling kernels.
- Theoretical guarantees for uniquely identifying FRI signals from modulo samples are missing for compactly supported kernels. Since identifiability results are independent of any algorithm, they can act as a benchmark to evaluate the efficiency of any algorithm.
- Existing algorithms for modulo sampling of FRI signals operate at a much higher sampling rate than the RoI. High sampling rates lead to expensive and power-consuming ADCs and require higher storage and computational costs. In addition, these methods require a higher number of measurements than the degrees of freedom of the FRI signals.
- Existing algorithms determine the true samples from their modulo ones up to a constant factor. This unknown factor is neglected in [40] during reconstruction. An approach to determine annihilating filter coefficients in the presence of the unknown factor was discussed in [41]. However, the approach will not generally work and require additional constraints, as discussed in this paper.

In this paper, we address these shortcomings. We consider the problem of modulo sampling of FRI signals by using a compactly supported SoS kernel. Our first objective is to derive identifiability results independent of any recovery algorithm and then develop an efficient algorithm in terms of sampling rate and the number of measurements. In this context, our main contributions are summarized as follows:

- We consider modulo sampling with a compactly supported SoS kernel for FRI signals. It is shown that the output of an SoS kernel is a PBL signal [7], [12]. The

PBL signal is folded by using a modulo operator and then sampled. We show that an L-order PBL signal can be uniquely identifiable up to a constant factor from their modulo samples if the number of samples per interval is greater than or equal to $2L+1$ and is prime. We also show that $2L+1$ samples are necessary. Due to the discrete nature of the Fourier spectra of PBL signals, the identifiability results of [13], which are derived for continuous spectra, are not applicable.

- To derive the FRI parameters from PBL samples (samples of the output of the SoS kernel), AF can be used. However, the samples are identifiable up to a constant factor, and AF cannot be directly applied. To address this issue, we consider three solutions. The first is oversampling, where $4L+1$ samples are considered in the first approach. In the second approach, only $2L+1$ samples are used by assuming that the time delays are on a grid. The third approach requires the amplitudes to be positive. This approach does not need oversampling or an on-grid assumption.
- We propose a sampling-efficient algorithm to estimate the FRI parameters from modulo samples. We use Itoh's approach [44] or the first-order difference approach [13] for unfolding the samples of the SoS kernel. For unfolding, we derive constraints on the SoS filter coefficients. We then establish bounds on the minimum sampling rate and the number of measurements for unfolding and FRI parameter estimation. In particular, we show that $2L+1$ modulo samples are sufficient for estimating the FRI parameters.
- We present simulation results where the proposed algorithm is compared with existing approaches [13], [15], [40]. We show that our method requires the lowest sampling rate and number of samples compared to [13], [15], and [40].

The paper is organized as follows. The next section presents the signal model and an explicit problem formulation. In Section III, we discuss the FRI sampling framework in the absence of modulo operation, which will be helpful in understanding the later sections. In Section IV, we present theoretical guarantees for the modulo-FRI framework. A practical algorithm is presented in Section V. The proposed algorithm is compared with existing results in Section VI, followed by conclusions.

II. PROBLEM STATEMENT

Consider a class of real-valued FRI signals $\mathcal{C}(h, L, a_{\max}, T_d)$ where each $f \in \mathcal{C}(h, L, a_{\max}, T_d)$ can be written as

$$f(t) = \sum_{\ell=1}^{L} a_\ell h(t - t_\ell), \tag{1}$$

where the FRI pulse $h(t)$ and the FRI parameters $\{a_\ell, t_\ell\}_{\ell=1}^{L}$ satisfy the following assumptions.

A1 The pulse $h(t)$ is known, real-valued, and time-limited to an interval $[0, T_h]$.

A2 The number of pulses L is known.

A3 For $\ell = 1, 2, \cdots, L$, $t_\ell \in (0, T_d] \subset \mathbb{R}$ for known T_d.

A4 The amplitudes are bounded by $a_{\max}$: $|a_\ell| \leq a_{\max}$, $\ell = 1, \cdots, L$.

The objective is to devise a reconstruction framework to recover $\{a_\ell, t_\ell\}_{\ell=1}^{L}$ from low-rate or sub-Nyquist samples. This is achieved by designing a sampling kernel $g(t)$ such that from a finite number of sub-Nyquist samples $y(nT_s)$ of the filtered FRI signal $y(t) = (f * g)(t)$, the FRI parameters $\{a_\ell, t_\ell\}_{\ell=1}^{L}$ are computed uniquely. We discuss the standard sampling and reconstruction approach in the next section.

In practice, the analog-to-digital converter (ADC) has a finite dynamic range, $[-\lambda, \lambda]$, and, typically, signals beyond this range are clipped before being sampled. Clipping results in loss of information. One way to overcome clipping is to map the signal to the dynamic range of the ADC. In this work, we consider a modulo operation on the analog signal to restrict to $[-\lambda, \lambda]$ before sampling (cf. Fig. 1). For any $a \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$, we define a modulo operation $\mathcal{M}_\lambda(\cdot)$ as

$$\mathcal{M}_\lambda(a) = (a + \lambda) \bmod 2\lambda - \lambda. \tag{2}$$

The modulo samples are denoted by $y_\lambda(nT_s) = \mathcal{M}_\lambda(y(nT_s))$. The objective is to design a sampling kernel and parameter estimation technique such that $\{a_\ell, t_\ell\}_{\ell=1}^{L}$ are determined from modulo samples $y_\lambda(nT_s)$ measured at a sub-Nyquist rate. In addition, it is desirable that the number of samples is close to the number of degrees of freedom (DoF) given by $2L$. In our formulation, we assume that the time-domain signals are real-valued. The framework could be extended to complex-valued signals by modifying the modulo-operator to fold both real and imaginary parts independently. In the next section, we discuss sampling and recovery in the absence of the modulo operation. The discussion will aid in deriving and comparing the results in the presence of modulo.

III. FRI SAMPLING AND RECOVERY WITHOUT MODULO

The FRI sampling and reconstruction problem is posed as a problem of designing a sampling kernel $g(t)$ such that from a finite number of filtered samples of $y(t) = (f * g)(t)$ one can determine the FRI parameters $\{a_\ell, t_\ell\}_{\ell=1}^{L}$. In particular, the goal is to recover the $2L$ unknowns from samples $\{y(nT_s)\}_{n=1}^{N}$ such that N is close to $2L$ and the sampling interval, T_s, is as large as possible.

The design of the sampling kernel is tightly coupled to the reconstruction strategy. Next, we discuss a widely used Fourier-domain recovery method and then examine the sampling kernel that enables this reconstruction [1], [7], [12].

A. FOURIER-DOMAIN RECONSTRUCTION

The Fourier transform of $f(t)$ is given as

$$F(\omega) = \int f(t) e^{-\mathrm{j}\omega t} \mathrm{d}t = H(\omega) \sum_{\ell=1}^{L} a_\ell \, e^{-\mathrm{j}\omega t_\ell}, \tag{3}$$

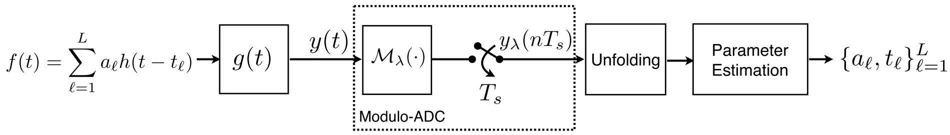


FIGURE 1. **Kernel-based FRI sampling and reconstruction approach: $g(t)$ denotes the sampling kernel; the modulo-ADC consists of a modulo folding operator followed by a conventional ADC with dynamic-range $[-\lambda, \lambda]$; T_s denotes the sampling rate.**

where $H(\omega)$ is the Fourier transform of $h(t)$. Next, consider the following samples

$$S(k\omega_0) = \frac{F(k\omega_0)}{H(k\omega_0)} = \sum_{\ell=1}^{L} a_\ell \, e^{-\mathrm{j}k\omega_0 t_\ell}, \quad k \in \mathcal{K}, \tag{4}$$

where $\mathcal{K}$ is a set integers and ω_0 is the sampling interval in the frequency domain. We assume[1] that $H(k\omega_0) \neq 0$ for $k \in \mathcal{K}$. It can be shown that $2L$ consecutive samples of $S(k\omega_0)$ are sufficient to uniquely identify $\{a_\ell, t_\ell\}_{\ell=1}^{L}$ if $\omega_0 = \frac{2\pi}{T_d}$ [1].

In practice, AF method can be used to determine $\{a_\ell, t_\ell\}_{\ell=1}^{L}$ from $\{S(k\omega_0)\}_{k\in\mathcal{K}}$ provided that $|\mathcal{K}| \geq 2L$. As the AF method plays a crucial role in designing an algorithm in the presence of modulo operation, we discuss the AF method at the end of this section.

Given that $2L$ Fourier samples are sufficient to identify the FRI parameters, the next question is how to compute the Fourier samples $\{F(k\omega_0)\}_{k\in\mathcal{K}}$ as $\{H(k\omega_0)\}_{k\in\mathcal{K}}$ are pre-computed from known $h(t)$. In the following, we discuss a sampling kernel-based approach to determine $\{F(k\omega_0)\}_{k\in\mathcal{K}}$ from sub-Nyquist samples of the filtered signal.

B. SAMPLING KERNEL AND SAMPLING

Let us consider a sum-of-sincs kernel [7] with impulse response

$$g(t) = \mathrm{rect}\left(\frac{t}{T_g}\right) \sum_{k\in\mathcal{K}} c_k e^{\mathrm{j}k\omega_0 t}, \tag{5}$$

where $\mathrm{rect}\left(\frac{t}{T_g}\right) = 1$ for $t \in [0, T_g]$ or zero otherwise. To keep the filter response and its output real-valued, we choose $\mathcal{K}$ as

$$\mathcal{K} = \{-K, \cdots, K\}, \quad \text{where} \quad K \geq L, \tag{6}$$

and $c_k = c_{-k}^*$. The choice of the set $\mathcal{K}$ implies that $|\mathcal{K}| \geq 2L + 1$, that is, one sample more than the DoF $2L$. For $T_g > T_h + T_d$, it can be shown that (cf. [12], [45]) the filtered signal $y(t) = (f * g)(t)$ is given as

$$y(t) = \sum_{k\in\mathcal{K}} c_k e^{\mathrm{j}k\omega_0 t} F(k\omega_0) \quad \text{for} \quad t \in T_{\mathrm{obs}}, \tag{7}$$

where $T_{\mathrm{obs}} = [T_h + T_d, T_g]$.

The filtered signal in T_{obs} is a linear combination of $\mathcal{K}$ Fourier samples of the input signal $f(t)$. Upon uniform sampling of $y(t)$ within the interval T_{obs} we have

$$y(nT_s) = \sum_{k\in\mathcal{K}} c_k e^{\mathrm{j}k\omega_0 nT_s} F(k\omega_0), \quad n \in \mathcal{N}, \tag{8}$$

where T_s is sampling interval and the set of sampling indices $\mathcal{N}$ is given by

$$\mathcal{N} = \{N_{\min}, \cdots, N_{\max}\},$$
$$\text{where } N_{\min} = \left\lceil \frac{T_h + T_d}{T_s} \right\rceil, \; N_{\max} = \left\lfloor \frac{T_g}{T_s} \right\rfloor. \tag{9}$$

The relation in (8) denotes a set of $|\mathcal{N}|$ linear equations with $|\mathcal{K}|$ unknowns. To compute $|\mathcal{K}|$ Fourier samples it is necessary to have $|\mathcal{N}| \geq |\mathcal{K}|$ time-domain samples which can be ensured by setting $T_g \geq T_h + T_d + |\mathcal{N}|T_s$. In addition, to uniquely determine $\{F(k\omega_0)\}_{k\in\mathcal{K}}$ from the samples $\{y(nT_s)\}_{n\in\mathcal{N}}$, it is necessary to ensure the elements of the set $\{e^{\mathrm{j}k\omega_0 T_s}\}_{k\in\mathcal{K}}$ are distinct. This condition is satisfied if we have $|\mathcal{K}|\omega_0 T_s \leq 2\pi$, that is, $T_s \leq \frac{T_d}{|\mathcal{K}|}$. To summarize, the FRI parameters can be computed from the samples $\{y(nT_s)\}_{N_{\min}}^{N_{\max}}$ provided that $T_g \geq T_h + T_d + |\mathcal{N}|T_s$ and

$$T_s \leq \frac{T_d}{|\mathcal{K}|}. \tag{10}$$

Since $|\mathcal{K}| \geq 2L$, the above result implies that the optimal sampling interval is $T_{s,opt} = \frac{T_d}{2L}$ and RoI is $\frac{2L}{T_d}$. The minimum number of time samples required to identify the FRI parameters is $2L$, which is equal to the FRI signal's DoF. The results are summarized in the following theorem [7], [12].

Theorem 1 (Sampling and Reconstruction of FRI Signals): Consider FRI signals in (1) that satisfy assumptions (A1)-(A3). The FRI parameters can be uniquely recovered from the filtered samples $\{y(nT_s) = (f * g)(nT_s)\}_{n\in\mathcal{N}}$ where the impulse response of the filters $g(t)$ is given as in (5) and the sampling set $\mathcal{N}$ as in (9), provided that $|\mathcal{N}| \geq |\mathcal{K}| \geq 2L$ and $T_s \leq \frac{T_d}{|\mathcal{K}|}$.

The FRI signals have finite support and finite degrees of freedom over the support. Hence, the conditions for perfect recovery are stated both in terms of the sampling rate and the number of samples. This fact plays an important role in the recovery algorithm of the modulo-based sampling scheme.

C. ANNIHILATING FILTER (AF) APPROACH

Define a discrete sequence as

$$s[k] = S(k\omega_0) = \frac{F(k\omega_0)}{H(k\omega_0)} = \sum_{\ell=1}^{L} a_\ell \, e^{-\mathrm{j}k\omega_0 \tau_\ell}, \quad k \in \mathcal{K}. \tag{11}$$

[1]The assumption that the spectral samples of the pulse $h(t)$ are non-vanishing is, typically, satisfied by many practically applied pulses due to their small time-support and large bandwidth.

Consider a $L+1$-length sequence $\{x[0], x[1], \cdots, x[L]\}$ and let the convolution $(s * x)[k]$ for $L-K \leq k \leq K$ be

$$(s * x)[k] = \sum_{\ell=1}^{L} a_\ell X(e^{\mathrm{j}\omega_0 \tau_\ell}) e^{-\mathrm{j}k\omega_0\tau_\ell}, \tag{12}$$

where $X(z)$ is the z-transform of the sequence $x[k]$. Consider determining the sequence $x[k]$ such that $(s*x)[k] = 0, L-K \leq k \leq K$, that is, a filter that annihilates the sequence $s[k]$. From (12), we rewrite the annihilation problem as $\mathbf{V}\tilde{\mathbf{x}} = \mathbf{0}$ where $\mathbf{V}$ is a Vandermonde matrix of size $(2K-L+1) \times L$ given by

$$\begin{pmatrix} e^{-\mathrm{j}\omega_0(L-K)\tau_1} & e^{-\mathrm{j}\omega_0(L-K)\tau_2} & \cdots & e^{-\mathrm{j}\omega_0(L-K)\tau_L} \\ e^{-\mathrm{j}\omega_0(L-K+1)\tau_1} & e^{-\mathrm{j}\omega_0(L-K+1)\tau_2} & \cdots & e^{-\mathrm{j}\omega_0(L-K+1)\tau_L} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\mathrm{j}\omega_0 K\tau_1} & e^{-\mathrm{j}\omega_0 K\tau_2} & \cdots & e^{-\mathrm{j}\omega_0 K\tau_L} \end{pmatrix} \tag{13}$$

and $\tilde{\mathbf{x}} = [a_1 X(e^{\mathrm{j}\omega_0\tau_1})\ a_2 X(e^{\mathrm{j}\omega_0\tau_2}) \cdots a_L X(e^{\mathrm{j}\omega_0\tau_L})]^{\mathrm{T}}$. Since τ_ℓs are distinct, for $K \geq L$, the matrix $\mathbf{V}$ has full column rank. Hence (13) holds if and only if $X(e^{\mathrm{j}\omega_0\tau_\ell}) = 0$ for $\ell = 1, 2, \cdots, L$, that is, if the zeros of Fourier transform of $x[k]$ are located at $\{e^{\mathrm{j}\omega_0\tau_\ell}\}_{\ell=1}^{L}$. Since $x[k]$ is a sequence of length $L+1$, it has L zeros, and there exists a unique sequence $x[k]$ that annihilates $s[k]$. Once the annihilating filter is determined, $\{t_\ell\}_{\ell=1}^{L}$ are estimated from its zeros. The annihilating filter $x[k]$ is determined from the sequence $s[k]$ by rewriting $(s*x)[k], L-K \leq k \leq K$ as a set of homogeneous equations $\mathbf{S}_T\mathbf{x} = \mathbf{0}$ where

$$\mathbf{S}_T = \begin{pmatrix} s[L-K] & s[L-K-1] & \cdots & s[-K] \\ \vdots & \vdots & \ddots & \vdots \\ s[0] & s[-1] & \cdots & s[-L] \\ \vdots & \vdots & \ddots & \vdots \\ s[K] & s[K-1] & \cdots & s[K-L] \end{pmatrix} \tag{14}$$

and $\mathbf{x} = [x[0]\ x[1]\ \cdots x[L]]^{\mathrm{T}}$. The Toeplitz matrix $\mathbf{S}_T \in \mathbb{C}^{(2K-L+1)\times(L+1)}$ is rank deficient, specifically, for $K \geq L$, it can be shown that $\mathrm{Rank}(\mathbf{S}_T) = L$ for a sequence $s[k]$ that consists of linear combinations of L complex exponentials as shown in (11) [1, Proposition15.2]. Hence, $\mathbf{S}_T$ has a unique non-zero null space vector $\mathbf{x}$. Once $\{\tau_\ell\}_{\ell=1}^{L}$ are determined, $\{a_\ell\}_{\ell=1}^{L}$ are found by solving set of linear equations (11).

IV. IDENTIFIABILITY RESULTS WITH MODULO

We now present identifiability results for modulo sampling of FRI signals. In the presence of a modulo operation (cf. Fig. 1), the samples are decomposed as

$$y_\lambda(nT_s) = y(nT_s) + z(nT_s), \tag{15}$$

where the values of the sequence $z(nT_s)$ are integer multiples of 2λ. The sequence $z(nT_s)$ is a function of $y(t)$, the input to the modulo block, and ensure that $|y_\lambda(nT_s)| \leq \lambda$. The samples $y_\lambda(nT_s)$ are functions of the FRI parameters. The question we would like to answer is whether the FRI parameters can be uniquely identifiable from $y_\lambda(nT_s)$.

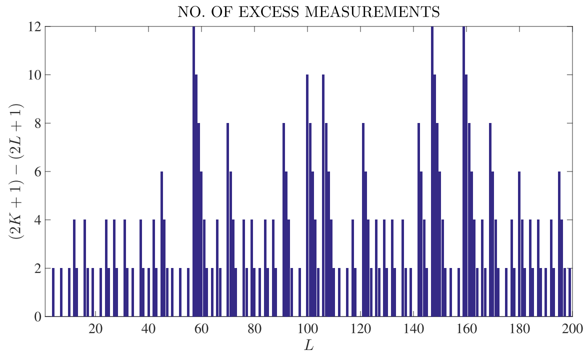


FIGURE 2. Number of excessive samples required above the optimum $2L+1$ with the prime condition.

As earlier, we use the SoS kernel (cf. (5)). Hence, $y(t)$ is a trigonometric polynomial as in (7). From Theorem 1 we note that $|\mathcal{N}| \geq |\mathcal{K}| \geq 2L$ samples of $y(t)$ are sufficient to uniquely identify the FRI signal provided that $T_s \leq \frac{T_d}{|\mathcal{K}|}$. Hence, we first present the identifiability results of the samples of a trigonometric polynomial from its modulo samples and then extend the results to the FRI case.

A. UNIQUENESS OF TRIGONOMETRIC POLYNOMIAL UNDER MODULO OPERATION

We first present results for a generic trigonometric polynomial and then show how it can be related to the FRI signal model. Consider a K-th order real-valued trigonometric polynomial as in (7). Let the polynomial be sampled at a rate $T_s = \frac{T_d}{2K'+1}$ where $K' \geq K$. For simplicity, we assume that $c_k = 1$ for $k \in \mathcal{K}$. The samples are given as

$$y(nT_s) = \sum_{k=-K}^{K} F(k\omega_0) e^{\mathrm{j}k\omega_0 nT_s}, \tag{16}$$

where $\omega_0 = \frac{2\pi}{T_d}$. Consider the problem of uniquely identifying $y(nT_s)$ from its modulo samples $y_\lambda(nT_s)$. In this case, our identifiability results are presented in the following theorem.

Theorem 2 (Identifiability of Trigonometric Polynomial From Modulo Samples): Consider the modulo samples $y_\lambda(nT_s)$ of the trigonometric polynomial (16), where $T_s = \frac{T_d}{2K'+1}$ with $K' \geq K$.

1) *If $K' \leq K$ then $y(nT_s)$ is not identifiable from $y_\lambda(nT_s)$.*
2) *If $K' > K$ and $2K'+1$ is prime then $y(nT_s)$ is uniquely identifiable up to a constant multiple of 2λ from its modulo samples $y_\lambda(nT_s)$.*

The proof is given in Appendix A.

In the FRI framework, Theorem 1 shows that from the filtered samples $y(nT_s)$ (as in (8)), the FRI signal can be uniquely recovered without modulo provided that $T_s \leq \frac{T_d}{|\mathcal{K}|}$ where $|\mathcal{K}| \geq 2L+1$. With modulo operation prior to sampling, we conclude that $y(nT_s)$ in (8) can be uniquely identifiable from its modulo samples up to a constant factor provided that $T_s \leq \frac{T_d}{|\mathcal{K}|}$, $K \geq L$, and the number of Fourier samples is prime, that is, $2K+1$ is prime. The prime condition on the number of Fourier samples, which is also equal to the number of time samples of $y(t)$ within an interval of length T_d, results in oversampling. For example, for $L = 4$, the optimum number of Fourier samples in the absence of modulo operation is 9, whereas, with modulo operation, it is 11, two samples more than the minimum. In Fig. 2, we plot the difference between the desired number of Fourier samples for unique identifiability and the optimum number of samples $2L+1$ for $2 \geq L \geq 200$. We observe that for $L \leq 56$, one is required to measure a maximum of 6 more samples than the optimal. For $57 \leq L \leq 200$, a maximum of 12 addition samples are required for certain values of L. In addition, for several values of L, no additional samples are required. For very large L, the next prime number could be far, as shown in the prime density theorem. However, in practice, the value of L is not very high. For example, in [46], $L = 80$ is used to model practical ultrasound signals. With this observation, we infer that the prime condition does not warrant high oversampling.

An alternative identifiability proof was recently derived in [47]. In the proof, the authors claimed that an oversampling factor of $3\left(1+\frac{1}{K}\right)$ is sufficient to uniquely recover the true samples of the trigonometric polynomial in (16) from the folded samples and $2K'+1$ need not be prime. Comparing these results with ours for FRI signals, we note that the result in [47] requires at least $3L$ more number of measurements, which is much higher than the current results.

Note that the identifiability results in Theorem 2 are different compared to that in [13] in terms of the class of the signal. In [13], the folded signal is a bandlimited signal with dense spectra. On the other hand, in Theorem 2, the signal is a periodic bandlimited signal with discrete spectra. Due to the discrete nature, one cannot use the line of proof in [13] here.

B. UNIQUENESS OF FRI SIGNAL UNDER MODULO OPERATION

In the previous section, we showed that trigonometric polynomials of the form (8) or (16) can be uniquely recovered from their modulo samples $y_\lambda(nT_s)$ up to a constant factor. In this section, we derive identifiability results for FRI signals from the recovered samples

$$\bar{y}(nT_s) = y(nT_s) + \beta, \quad n = N_{\min}, \cdots, N_{\max}-1, \tag{18}$$

where $\beta \in 2\lambda\mathbb{Z}$ is an unknown constant parameter.

By substituting $y(nT_s)$ from (8) into (18) we have

$$\bar{y}(nT_s) = \sum_{k\in\mathcal{K}} c_k e^{jk\omega_0 nT_s} F(k\omega_0) + \beta. \tag{19}$$

From the known $\{c_k\}_{k\in\mathcal{K}}$ and the above set of linear equations, we compute the Fourier samples $\{\bar{F}(k\omega_0)\}_{k\in\mathcal{K}}$ where

$$\bar{F}(k\omega_0) = \begin{cases} F(k\omega_0), & k \in \mathcal{K}\backslash\{0\}, \\ F(0)+\beta, & k = 0. \end{cases} \tag{20}$$

Here we assumed that there are a sufficient number of samples, that is, $N_{\max} - N_{\min} \geq |\mathcal{K}|$. As in (11), we construct a sequence $\bar{s}[k]$ defined as

$$\begin{aligned}\bar{s}[k] &\triangleq \frac{\bar{F}(k\omega_0)}{H(k\omega_0)} = s[k] + \bar{\beta}\,\delta[k] \\ &= \sum_{\ell=1}^{L} a_\ell\, e^{-jk\omega_0\tau_\ell} + \bar{\beta}\,\delta[k], \quad k \in \mathcal{K}, \end{aligned} \tag{21}$$

where $\delta[k]$ is the Kronecker impulse and $\bar{\beta} = \frac{\beta}{H(0)}$. When β is zero, one can uniquely identify the FRI parameters $\{a_\ell, \tau_\ell\}_{\ell=1}^{L}$ from $2L$ consecutive samples of $\bar{s}[k]$ (see Section III-C). However, for a nonzero β, uniqueness is not clear.

To have a unique solution, a possible approach is to neglect the sample at $k = 0$ and determine the FRI parameters from the remaining samples of $\bar{s}[k]$. With the missing zeroth sample, in general, uniqueness is not guaranteed when one considers an optimum number of Fourier measurements ($K = L$). See Appendix B for details. To address this issue, oversampling or constraints on the FRI parameters can be used, as discussed in the following.

1) OVERSAMPLING FOR UNIQUE IDENTIFIABILITY WITH MISSING SAMPLE

The FRI parameters can be uniquely determined from any $2L$ consecutive samples of $\bar{s}[k]$ that does not include the zeroth sample. This condition holds for $K \geq 2L$, and as a consequence annihilating filter can be applied to either the sequence $\{\bar{s}[k] : 1 \leq k \leq K\}$ or to the sequence $\{\bar{s}[k] : -K \leq k \leq -1\}$ to uniquely determine the time delays $\{\tau_\ell\}_{\ell=1}^{L}$. However, this approach requires twice the number of samples compared to the case when $\beta = 0$.

Next, we discuss approaches where oversampling is not required and constraints on the FRI parameters are used for identifiability. In this context, we first consider an existing approach (with the zeroth sample missing) where the time delays are assumed to be on a grid and then propose a method where the amplitudes are assumed to be positive, and the zeroth sample is taken into consideration.

2) ON-GRID TIME DELAYS WITH MISSING SAMPLE

In [48], the problem of recovering FRI parameters from Fourier measurements is considered where the zeroth frequency sample is not measured. For recovery of the parameters, the authors considered either $K \geq 2L$ measurements with double the sampling rate as discussed in the previous paragraph or assumed that the time delays are on a grid. In the latter case, with on-grid time delays, $K \geq L$ Fourier measurements, with missing zeroth Fourier measurement, are shown to be sufficient for recovering the FRI parameters.

3) POSITIVE AMPLITUDES CONSTRAINT WITHOUT A MISSING SAMPLE

While on-grid time delays are a strong assumption in a practical scenario, an alternative approach was presented in the early 1900s. In this approach, Carathéodory showed that a positive ($a_\ell > 0, \ell = 1, \cdots, L$) linear combination of L complex exponentials is uniquely determined from its zeroth sample and any other $2L$ samples [49]. The Carathéodory's result requires to have the zeroth sample, whereas we have ambiguity in this sample. In what follows we show that with the assumption $a_\ell > 0, \ell = 1, \cdots, L$ one can uniquely recover parameters $\{a_\ell, \tau_\ell\}_{\ell=1}^{L}$ even with an unknown ambiguity.

For $K = L$, the Toeplitz matrix $\bar{\mathbf{S}}$ (as in (14)) constructed from the sequence $\bar{s}[k]$ is related to true samples as in (17), as shown at the bottom of the page. It can be shown that Rank$(\mathbf{S}) = L$, and hence, the time delays can be determined uniquely from its null-space vector $\mathbf{x}$. However, the annihilating filter $\mathbf{x}$ is not necessarily in the null space of $\bar{\mathbf{S}}$. Despite that, we show that $\mathbf{x}$ can be uniquely determined from $\bar{\mathbf{S}}$ by using the fact that the amplitudes are positive. To this end, we rely on the following lemma.

Lemma 1: Let $\sigma(\bar{\mathbf{S}}), \mathcal{E}(\bar{\mathbf{S}})$ and $\sigma(\mathbf{S}), \mathcal{E}(\mathbf{S})$ denote pair of sets of eigenvalues and eigenvectors of matrices $\bar{\mathbf{S}}$ and $\mathbf{S}$, respectively. Then we have the same eigenvectors, that is, $\mathcal{E}(\bar{\mathbf{S}}) = \mathcal{E}(\mathbf{S})$, and $\sigma(\bar{\mathbf{S}}) = \bar{\beta} + \sigma(\mathbf{S})$.

The proof is deferred to Appendix C

Lemma 1 suggests that the annihilating filter $\mathbf{x}$, which is in $\mathcal{E}(\mathbf{S})$, is also an eigenvector of $\bar{\mathbf{S}}$. Specifically, we have $\bar{\mathbf{S}}\mathbf{x} = \bar{\beta}\mathbf{x}$. The question is how to uniquely identify $\mathbf{x}$ from the eigenvectors of $\bar{\mathbf{S}}$. To this end, we use the condition that the amplitudes of the FRI signal are positive. By using Carathéodory-Toeplitz theorem (see [1, Sec. 15.2.3], [50, Sec. 4.9.2]) it can be shown that $\mathbf{S}$ is positive semidefinite if $a_\ell > 0$. Hence, we have that $0 \in \sigma(\mathbf{S})$ as $\mathbf{S}\mathbf{x} = \mathbf{0}$ and the remaining eigenvalues are positive. Combining these results with those in Lemma 1, we conclude that $\bar{\beta}$ is the minimum eigenvalue of $\bar{\mathbf{S}}$. Hence, the eigenvector of $\bar{\mathbf{S}}$ associated with its minimum eigenvalue is the desired annihilating filter $\mathbf{x}$. Since the annihilating filter $\mathbf{x}$ is unique for a given set of FRI parameters, the FRI parameters can be uniquely determined from $\bar{s}[k]$.

A similar eigenvector-based approach was suggested in [41]. However, the approach does not assume that the amplitudes are positive, a condition without which we can distinguish the annihilating filter from the eigenvectors of matrix $\bar{\mathbf{S}}$.

We call this approach the ambiguous sample annihilating filter (ASAF) method. Note that the ambiguous sample need not be the zeroth one. Lemma 1 holds for any sum-of-exponential sequence consisting of L frequencies with positive amplitudes provided that the sample index set $\mathcal{K}$ has at least $2L + 1$ consecutive integers and $(L + 1)$-th sample is either missing or has measurement uncertainty. To verify our theory we consider $2L + 1$ uniform samples of sum-of-complex-exponential signal $s(t) = \sum_{\ell=1}^{L} a_\ell\, e^{\mathrm{j}\omega_\ell t}$ where $a_\ell > 0$. In this particular example, we set $L = 3$, $a_\ell = 1$, and frequencies as $2\pi[0.2, 0.4, 0.6]$. We consider the problem of recovering the frequencies and amplitudes from $s(nT_s)$, $n \in \{-L, \cdots, -1, 1, \cdots, L\}$ where $T_s = 1$. The real and imaginary values of the original signal $s(t)$ and estimated signal $\widehat{s}(t)$ are shown in Fig. 3. We observe perfect reconstruction using the ASAF method.

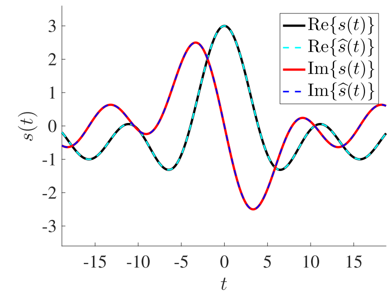


FIGURE 3. Reconstruction of sum-of-complex-exponential signal *s*(*t*) from its uniform samples with a missing sample. We observe perfect reconstruction by using the ASAF algorithm.

We summarize identifiability results for FRI signals from modulo samples in the following theorem.

Theorem 3: Consider an FRI signal $f(t)$ as in (1) that follows assumptions (A1) - (A3). Consider an SoS sampling kernel $g(t)$ as in (5) whose coefficients satisfy the inequality in (32) where $\mathcal{K} = \{-K, \cdots, K\}$ and λ is a known non-zero real number. Consider the modulo samples $\{y_\lambda(nT_s) = \mathcal{M}_\lambda((f * g)(nT_s))\}_{n\in\mathcal{N}}$, where $T_s \leq \frac{T_d}{2K'+1}$ and $K' \geq K \geq L$. The sample index set $\mathcal{N}$ is defined in (9) for $|\mathcal{N}| \geq 2K + 1$. Then

1) *FRI signal cannot be uniquely identifiable from the modulo samples if $K' \leq K$.*
2) *FRI signal is uniquely identifiable from the modulo samples if $K' > K$ and $2K' + 1$ is prime and one of the following holds*
   a) *$K \geq 2L$.*
   b) *The time delays are on a grid and $K \geq L$.*
   c) *The amplitudes are positive and $K \geq L$.*

To understand the results of Theorem 3 in comparison to Theorem 1 (no modulo folding), we consider the process of determining the FRI parameters from the modulo samples as shown in Fig. 4. The process has three steps: (1) Unfolding, (2) Determining the Fourier samples from the unfolded or true samples, and (3) Estimating the FRI parameters from the

$$\underbrace{\begin{pmatrix} \bar{s}[0] & \bar{s}[-1] & \dots & \bar{s}[-L] \\ \bar{s}[1] & \bar{s}[0] & \dots & \bar{s}[-L+1] \\ \vdots & \vdots & \ddots & \vdots \\ \bar{s}[L] & \bar{s}[L-1] & \dots & \bar{s}[0] \end{pmatrix}}_{\bar{\mathbf{S}}} = \underbrace{\begin{pmatrix} s[0] & s[-1] & \dots & s[-L] \\ s[1] & s[0] & \dots & s[-L+1] \\ \vdots & \vdots & \ddots & \vdots \\ s[L] & s[L-1] & \dots & s[0] \end{pmatrix}}_{\mathbf{S}} + \bar{\beta} \underbrace{\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}}_{\mathbf{I}_{L+1}} \tag{17}$$

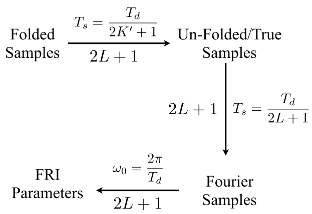


FIGURE 4. Flow diagram of FRI parameter estimation from folded samples: Sampling intervals and the number of samples for each step are added. For sufficient condition we need $K' \geq L$ and $2K'+1$ should be prime.

Fourier samples. In Section I, we combined steps 2 and 3 as FRI recovery for ease of explanation, but here, we separated them to have a better understanding. In the process shown in Fig. 4, we assumed that the time delays are on a grid or the amplitudes are positive. We added the maximum sampling intervals and a minimum number of samples required at each step. Note that the parameters, sampling interval, and number of samples are crucial for unique recovery as we deal with compactly supported signals with finite DoF. For example, to uniquely determine the Fourier measurements $\{F(k\omega_0)\}_{k=-K}^{K}$ from the samples $y(nT_s)$ in (16), we need a minimum of $2K+1$ samples with maximum sampling interval $T_s = \frac{T_d}{(2K+1)}$.

Both without folding and with folding approaches require a minimum of $2L+1$ Fourier measurements to determine the FRI parameters uniquely. In both scenarios, the sampling interval in the Fourier domain is $\omega_0 = \frac{2\pi}{T_d}$. Next, to compute the Fourier measurements from the unfolded or true time samples, a minimum of $2L+1$ time samples are required with sampling interval $T_s = \frac{T_s}{2L+1}$. Up to this point, both approaches are identical in terms of the number of samples and sampling rates. In case of modulo sampling, to unfold $2L+1$ samples, $2L+1$ folded samples are sufficient but the sampling interval is $T_s = \frac{T_s}{2K'+1}$ where $K' \geq L$ and $2K'+1$ is prime. In terms of the number of samples, all the steps require $2L+1$ samples, which is optimum. The sampling intervals in steps 2 and 3 in both approaches too are optimum. However, in modulo sampling, unfolding may require oversampling for some values of L. Specifically, oversampling is not required for the values of L for which $2L+1$ is prime. In this case $K' = L$.

The identifiability results in Theorem 3 show that without any constraints on the FRI parameters, one requires twice the number of samples. Whereas, with constraints either on the amplitudes or time delays, identifiability is achieved with the optimal number of samples. Unlike the identifiability results for bandlimited signals [13] and periodic bandlimited signals, FRI signals are uniquely identifiable without any constant additive factor. Next, we discuss a practical algorithm to recover FRI parameters from the modulo samples.

V. FILTER DESIGN AND RECOVERY ALGORITHM

In this section, we propose an algorithm to estimate the FRI parameters from the modulo samples. As discussed in the previous section, we consider an SoS kernel-based modulo-sampling framework. Our algorithm follows a two-stage approach where, in the first stage, unfolding is performed, and in the second stage, FRI parameters are estimated from the unfolded samples. For unfolding, we apply a first-order difference approach. The signal reconstruction step is based on the ASAF algorithm. The unfolding step depends on the variation of the filtered signal, and hence, it strongly depends on the choice of the SoS filter coefficients. In the following, we first discuss the unfolding and reconstruction and then present an approach to design the SoS filter.

A. UNFOLDING AND SIGNAL RECONSTRUCTION

Our unfolding approach is based on the first-order difference [13], [44]. The key steps in this method are stated as follows. The modulo samples can be decomposed as

$$y_\lambda(nT_s) = y(nT_s) + z(nT_s), \tag{22}$$

where the values of the sequence $z(nT_s) \in 2\lambda\mathbb{Z}$. If the true samples satisfy the inequality

$$|y((n+1)T_s) - y(nT_s)| \leq \lambda \tag{23}$$

then the following equality holds

$$\begin{aligned} d(nT_s) &\triangleq \mathcal{M}_\lambda\left(y_\lambda((n+1)T_s) - y_\lambda(nT_s)\right), \\ &= y((n+1)T_s) - y(nT_s), \end{aligned} \tag{24}$$

for $n = N_{\min}, \cdots, N_{\max} - 1$. This implies that the difference of true samples $y((n+1)T_s) - y(nT_s)$ can be determined from the modulo samples provided that (23) holds. From the differences, the true samples, $y(nT_s)$, are determined up to an unknown constant factor. We can use the annihilating filter approach discussed in Section IV-B to determine the time delays and amplitudes of the FRI signal. Hence, for unfolding and FRI recovery, we must ensure that (24) is satisfied. Note that $y(t)$ is a function of the FRI signal and the sampling kernel. Since one cannot modify the FRI signal, we discuss an approach to design the sampling kernel such that (24) holds.

B. FILTER DESIGN

For the filter design problem, we consider modifying the SoS filter given in (5). To this end, we first derive a general condition on the filter's impulse response and then discuss the design of the SoS filter coefficients such that the resulting filter satisfies the conditions.

Recall that

$$y(t) = \sum_{\ell=1}^{L} a_\ell\, (h*g)(t - t_\ell) \tag{25}$$

and $(h*g)(t) = \displaystyle\int h(\tau)g(t-\tau)\mathrm{d}\tau$. Therefore

$$\begin{aligned} &|y((n+1)T_s) - y(nT_s)| \\ &=\leq \sum_{\ell=1}^{L} |a_\ell|\, \left|(h*g)((n+1)T_s - t_\ell) - (h*g)(nT_s - t_\ell)\right|, \end{aligned}$$

$$\leq \int |h(\tau)| \left(|g((n+1)T_s - \tau) - g(nT_s - \tau)|\right) \mathrm{d}\tau \sum_{\ell=1}^{L} |a_\ell|,$$

$$\leq La_{\max} \mathcal{L}_g(T_s) \int |h(\tau)| \mathrm{d}\tau, \tag{26}$$

where $|g((n+1)T_s - \tau) - g(nT_s - \tau)| \leq \mathcal{L}_g(T_s)$ and $\mathcal{L}_g(T_s)$ is the Lipschitz constant for $g(t)$. Since $L, a_{\max}$ and $\alpha_h = \int |h(\tau)|\mathrm{d}\tau$ are known *a priori*, if the filter $g(t)$ is designed such that for a desired T_s we have

$$\mathcal{L}_g(T_s) \leq \frac{\lambda}{La_{\max}\alpha_h}, \tag{27}$$

then (23) holds and hence we can estimate $y(nT_s)$ from the modulo samples up to a constant factor.

Next, consider an SoS filter with impulse response as in (5). The filter is parameterized by the coefficients $\{c_k\}_{k\in\mathcal{K}}$ with a constraint that $c_k = c^*_{-k}$. We derive the conditions on the coefficients such that the resulting SoS filter follows (26). For this, the Lipschitz constant $\mathcal{L}_g(T_s)$ is computed by using the mean value theorem. For every t and $t+T_s$ within the support of $g(t)$, that is, in the interval $[0, T_g]$ we have that

$$|g(t+T_s) - g(t)| \leq \sup_{\tau\in[0,T_g]} g'(\tau)\, T_s \tag{28}$$

$$\leq 2\omega_0 T_s \sum_{k=1}^{K} k|c_k| = \mathcal{L}_g(T_s). \tag{29}$$

While deriving the previous inequality, we used the conjugate symmetry constraint $c_k = c^*_{-k}$.

Substituting (29) in (26), we note that (24) holds if the filter coefficients and the sampling interval satisfy the following condition:

$$2La_{\max}\alpha_h\omega_0 T_s \left(\sum_{k=1}^{K} k|c_k|\right) \leq \lambda. \tag{30}$$

Specifically, if the condition is satisfied, then the samples can be perfectly unfolded up to a constant factor.

Next, consider the optimal sampling interval that corresponds to the minimum possible sampling rate in the absence of modulo operation:

$$T_{\text{opt}} = \frac{T_d}{(2K+1)}. \tag{31}$$

If one wants to operate at the minimum possible sampling rate, that is, $T_s = T_{\text{opt}}$ then the filter coefficients must satisfy the inequality

$$\sum_{k=1}^{K} k|c_k| \leq \frac{\lambda}{2\omega_0 T_s La_{\max}\alpha_h} = \frac{(2K+1)\lambda}{4\pi La_{\max}\alpha_h}. \tag{32}$$

Alternatively, if the filter coefficients are designed to satisfy a particular criterion, then one may have to sample above the minimum sampling rate. For example, in the presence of noise, the coefficients could be optimally designed to minimize reconstruction accuracy as in [7].

Next, to complete the discussion on the filter design, we consider a simple choice of filter coefficients. Consider

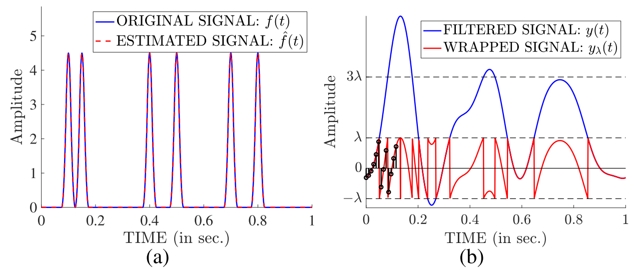


FIGURE 5. Reconstruction of FRI signal from modulo samples: (a) FRI signal $f(t)$ and its reconstruction $\hat{f}(t)$ where $f(t)$ consists of $L = 6$ pulses with $a_{\max} = 6$ and $T_d = 1$; (b) Filtered signal $y(t)$ and its wrapped counterpart $y_\lambda(t)$ where $c_k = 1$ and $\lambda = 0.2\|y(t)\|_\infty$. The signal is reconstructed by using ASAF where OF = 8 and $2L+1$ samples (shown in black) are used for recovery.

an SoS filter with

$$c_k = 1, \quad k = -K, \cdots, K. \tag{33}$$

The dynamic range of $y(t)$ is a function of the coefficients. The maximum value of the filtered output, $\|y(t)\|_\infty$, in terms of the filter coefficients and the FRI parameters, is derived in the following.

Using Young's inequality[2]

$$\|(h*g)(t)\|_\infty \leq \|h(t)\|_1 \, \|g(t)\|_\infty \leq \alpha_h \left(|c_0| + 2\sum_{k=1}^{K}|c_k|\right). \tag{34}$$

Next, consider the following relations

$$|y(t)| = \left|\sum_{\ell=1}^{L} a_\ell\, (h*g)(t-t_\ell)\right| \leq La_{\max}\, |(h*g)(t)|$$

$$\leq La_{\max}\alpha_h \left(|c_0| + 2\sum_{k=1}^{K}|c_k|\right) = \|y(t)\|_\infty, \tag{35}$$

where the last inequality is derived by using (34). Hence for $c_k = 1$, we have that

$$\|y(t)\|_\infty = La_{\max}\alpha_h(2K+1). \tag{36}$$

Let the sampling interval with modulo be

$$T_s = \frac{T_{\text{opt}}}{\text{OF}}, \tag{37}$$

where OF is the oversampling factor. Substituting (33), (37), and (36) in (30) we have

$$\text{OF} \geq 2\pi \frac{La_{\max}\alpha_h}{\lambda} \frac{K(K+1)}{(2K+1)} \approx \frac{\pi}{2} \frac{\|y(t)\|_\infty}{\lambda}, \tag{38}$$

for $\|y(t)\|_\infty > \lambda$. For $\|y(t)\|_\infty < \lambda$, we need not over sample, that is, OF = 1.

To summarize, if all the SoS kernel coefficients are set to be unity, then by oversampling by a factor computed using (38), one can ensure that (23) is satisfied and subsequently, the

[2] Suppose $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, and $p^{-1} + q^{-1} = r^{-1} + 1$ with $1 \leq p, q \leq r \leq \infty$, then $\|f*g\|_r \leq \|f\|_p\|g\|_q$.

FRI parameters can be estimated using ASAF. To illustrate the accuracy of the algorithm we consider recovery of an FRI signal consisting of $L = 6$ pulses with $a_{\max} = 6$ and $T_d = 1$. The pulses $h(t)$ are third-order β-splines. The signal is filtered by using the SoS filter with $c_k = 1$. The filtered signal is wrapped by using $\lambda = 0.2\|y(t)\|_\infty$ and then sampled with an OF = 8 for $K = L$. The FRI parameters are recovered by using the ASAF algorithm with $2L + 1$ samples of $y_\lambda(t)$. In Fig. 5(a), we show that the FRI signal is perfectly recovered. The filtered signal and its modulo version are shown in Fig. 5(b) for visualization. In Fig. 6, we showed perfect reconstruction for $\lambda = 0.1\|y(t)\|_\infty$ and $\lambda = 0.05\|y(t)\|_\infty$.

VI. A COMPARISON OF ALGORITHMS

In this section, we compare the proposed algorithm with those in [13], [14], [15], and [40]. The algorithms in [13] and [14] consider an ideal LPF as a sampling kernel. While the unwrapping in [13] requires almost 17 times oversampling, unwrapping in [14] needs to oversample above the RoI. However, simulation results were not presented for the LP approach in [14], and, as we show later, in practice, the convergence of the algorithm requires a significantly higher sampling rate. For the rest of the discussion, the algorithms in [13], and [14] will be referred to as higher-order-difference (HoD) and linear-prediction (LP), respectively.

In both the algorithms mentioned above, one is required to use all the countably infinite unwrapped samples to determine the Fourier samples of the FRI signals for subsequent processing. In [40], Bhandari et al. considered sampling and *local* reconstruction of FRI signals by using a periodic lowpass filter instead of an ideal LPF. Due to the periodic bandlimited nature of the signal, a finite number of samples measured over a period are sufficient to determine the Fourier samples. Specifically, in [40], the number of time samples and the sampling rate are $2L + 1 + \frac{\|y\|_\infty}{\lambda}$ and $\frac{2\pi e(2K+1)}{T_d}$, respectively, where $K \geq L$. The oversampling factor is $2\pi e$, which is approximately 17 times higher than the RoI. In [40], the number of samples is inversely proportional to λ, and many measurements are required for low dynamic range ADCs. In contrast, the sampling rate is independent of the dynamic range of the ADC and always operates at 17 times the RoI. In addition, the authors do not discuss the ambiguity issue that arises while recovering the FRI signals from the Fourier samples. As the method uses local reconstruction together with HoD, we refer to it as local-HoD or (L-HoD).

In [15] and [41], the authors showed that the trigonometric polynomial or periodic bandlimited (PBL) signals as in (16) (or (8)) could be uniquely recovered (up to a constant factor) from its modulo samples provided that $T_s \leq \frac{T_d}{2(K+M+1)}$ where M denotes the number of folding instants incurred in an interval of length T_d while wrapping $y(t)$. All the samples within the one-time interval, $2(K + M + 1)$, are used for recovering $y(nT_s)$ from its modulo samples. The number of folding instants M is a function of $\frac{\|y(t)\|_\infty}{\lambda}$, however, an upper bound on M is not derived in [15]. For a better comparison, we derived an upper bound in the Appendix. In the following, the approach in [15] is referred to as PBL.

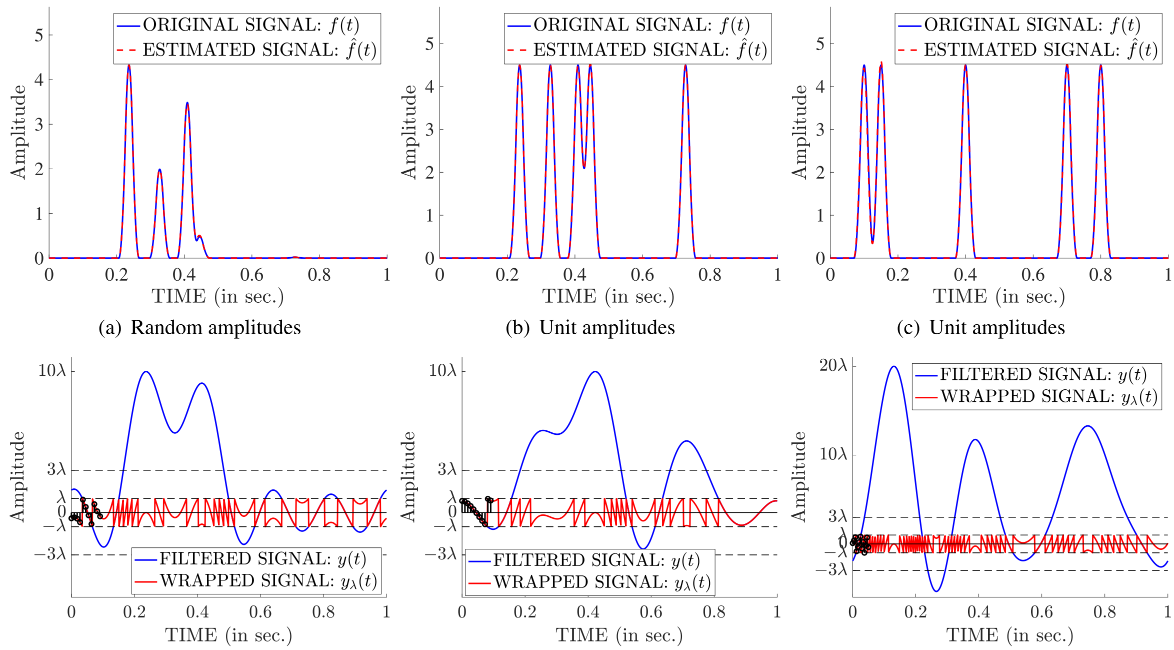
The method proposed in [41], which we refer to as Fourier-Prony or FP, is also based on the unfolding approach in [15]. Unlike [15], which is focused on PBL signals, [41] considers FRI signals which are lowpass filtered to have PBL structure. Since unfolding operation results in ambiguity at the zeroth frequency, the author in [41] suggested an eigenvector-based approach, which is similar to that discussed in Section IV-B.3. However, the positive-amplitudes assumption was not made in their approach which will not lead to perfect recovery.

A comparison of the algorithms is summarized in Table 1. In these comparisons, we consider the SoS filter with all unit coefficients. We have specified the worst-case sampling rates and the number of measurements for each method. In practice, the algorithms may operate at a lower rate. For example, in [41], the authors do not use the upper bound on M and suggest an approach to estimate it from the samples.

We observe that our algorithm operates with the lowest number of measurements, whereas, in the other approaches, either all countably infinite measurements are required [13], [14], or they are inversely proportional to the dynamic range of the ADC [15], [40]. Comparing the sampling rates, we observe that for recovering periodic BL signals as in [15] and [41] and in our approach, the rate is inversely proportional to λ, whereas, for BL signals, it is independent of λ. For better comparison, sampling rates for different algorithms are shown in Fig. 7 where we note that the LP approach [14] has the lowest possible rate but requires an infinite number of samples. On the other hand, the proposed approach is second best in terms of sampling rate and requires the lowest number of samples.

Note that the approaches in [13], [14], and [15] are not explicitly designed for FRI signal recovery, which is indicated in the last row of the table. However, while applying these methods for FRI signals, one can use the proposed ASAF approach for recovery.

Next, we compare the algorithms in terms of their accuracy in the estimation of the FRI parameters. Specifically, we consider the mean-squared-error (MSE) in the estimation of time delays which is computed as $\text{MSE} = \sum_{\ell=1}^{L} |\tau_\ell - \hat{\tau}_\ell|^2$ where $\hat{\tau}_\ell$ is an estimate of τ. First, we compare the algorithms for different values of λ without noise. In these simulations, we consider $L = 3$, $T_d = 1$, and $h(t)$ to be the Dirac impulse. The time delays are generated uniformly at random over the interval (0, 1] and are kept constant throughout the experiments. The amplitudes are set to be unity to ensure maximum variation in the filtered signal. The filtered signals are normalized to have a maximum unity amplitude prior to the modulo operation. After unwrapping, we apply ASAF to estimate the time delays. The MSEs in the estimations are shown in Fig 8(a). We observe that the proposed algorithm can estimate the time delays up to machine precision by using seven samples compared to 239, 120, 43, and 109 samples used by HoD, LP, L-HoD, and FP algorithms. Ideally, both



(a) Random amplitudes (b) Unit amplitudes (c) Unit amplitudes

(d) $\|y(t)\|_\infty/\lambda = 10$, OF = 12 (e) $\|y(t)\|_\infty/\lambda = 10$, OF = 12 (f) $\|y(t)\|_\infty/\lambda = 20$, OF = 22

FIGURE 6. FRI signal reconstruction from modulo samples for $L = 5$, $T_d = 1$ sec., $a_{\max} = 6$, $h(t)$ is third-order b-spline: For (a), amplitudes are chosen randomly; For (b) and (c) amplitudes are equal to $a_{\max}$; time-delays are generated uniformly at random between $(0, T_s)$; The samples used for recovery are shown in black; We observe perfect reconstruction for $\|y(t)\|_\infty/\lambda = 10, 20$, with sufficient oversampling.

TABLE 1. A Comparison of the Proposed (With positive Amplitudes) Sampling and Reconstruction Method with that in [13], [14], [15], [40], and [41].

	L-HoD [40]	PBL [15], FP [41]	HoD [13]	LP [14]	Our Approach
	Sampling Setup				
Sampling Kernel	Periodic LPF (Impractical)	Periodic LPF/Compact SoS filter	Ideal LPF (Impractical)	Ideal LPF (Impractical)	Compact SoS, practically feasible
Minimum Sampling Rate	$\frac{2\pi e(2L+1)}{T_d}$	$\frac{(2L+1)}{T_d} + \frac{4L}{T_d}\frac{\|y(t)\|_\infty}{\lambda}$	$\frac{2\pi e(2L+1)}{T_d}$	$> \frac{2L}{T_d}$	$\frac{\pi}{2}\frac{\|y(t)\|_\infty}{\lambda}\frac{(2L+1)}{T_d}$
	Unfolding and Reconstruction in the Absence of Noise				
Min. Number of Samples	$2L+1+7\frac{\|y\|_\infty}{\lambda}$	$2L+1+4L\frac{\|y(t)\|_\infty}{\lambda}$	All countably infinite samples	All countably infinite samples	$2L+1$
Unfolding approach	Higher-order differences	Out-of-band energy	Higher-order differences	Linear prediction with Chebyshev coefficients	First-order difference
FRI Reconstruction	Annhilating filter ignored the constant factor	Not applicable for PBL, Annihilating filter for FP; Issue with the constant factor	Not applicable	Not applicable	Modified AF including the constant factor

HoD and LP methods require an infinite number of samples. However, in practice, one can use a finite number of samples and the truncation results in a larger error in these algorithms. Both HoD and L-HoD operate at 17 times the rate of innovation while LP is operating with OF = 3. For the FP method, we observe that the maximum number of firings was $M = 50$, and hence we use OF = 15.5. The sampling rate of the proposed algorithm varies with λ, and in this particular example OF = 7.7 for $\lambda = 0.2$ and OF = 1.7 for $\lambda = 0.9$.

To evaluate the performance of the algorithms in the presence of noise, we added a zero-mean Gaussian noise to the modulo samples prior to unwrapping. In Fig. 8(b), we show the MSEs, which are averaged over 1000 independent realizations of noise samples for different signal-to-noise ratios (SNRs) for $\lambda = 0.2$. Due to the unbounded nature of the noise, the unwrapping algorithms required a higher sampling rate for convergence. In these simulations, we use OF = 7 for the proposed algorithm and OF = 17 for the rest of the three methods. The number of samples used in the proposed approach is 50, whereas 120 samples were used in L-HoD. Both HoD and LP considered 239 samples. As SNR increases, the errors in the proposed method and

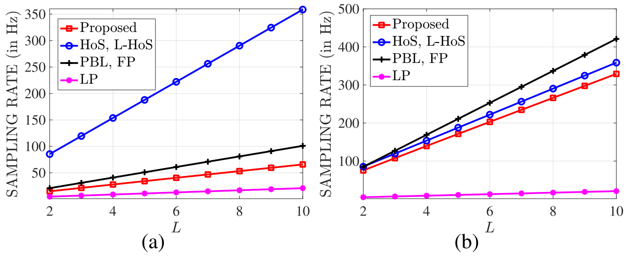


FIGURE 7. Sampling rate comparisons for different methods: (a) $\|y(t)\|_\infty/\lambda = 2$ and (b) $\|y(t)\|_\infty/\lambda = 10$.

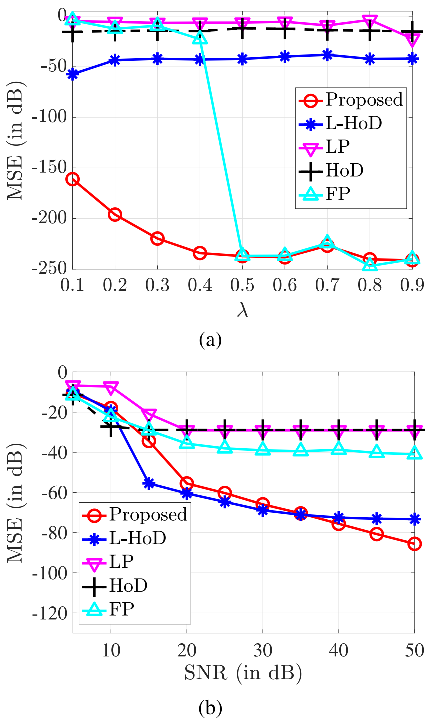


FIGURE 8. Comparison of different methods for $L = 3$: (a) MSEs in the estimation of time-delays in the absence of noise; (b) MSEs in the estimation of time-delays for different noise levels for $\lambda = 0.3\|y(t)\|_\infty$. In the absence of noise, the proposed methods result in the lowest error. In the presence of noise, the numbers of samples and OF for each method are given as (i) HoD: 239 samples, OF = 17, (ii) LP: 120 samples, OF = 3, (iii) L-HoD: 43 samples, OF = 17, (iV) FP: 109 samples, OF = 15.5, and (V) Proposed: 7 samples, OF = 7.7.

L-HoD decrease, whereas they stagnate for HoD and LP at higher SNRs due to truncation issues. The error in the FP method is lower compared to the LP ad HoD approach, but it is higher than MSEs in L-HoD and proposed methods. For SNRs between 10 to 30 dB, the L-HoD method has a lower error than the proposed algorithm but at the cost of a higher sampling rate and the number of samples.

For time-delay estimation, we are unable to attain a reasonable MSE while using the algorithm presented in [15], and hence the corresponding results were not included.

VII. MODULO-FRI ON A HARDWARE PROTOTYPE

In this section, we show that we can estimate FRI parameters by using the proposed annihilating filter with a missing Fourier sample. There are several works that discussed hardware prototypes for modulo sampling and their

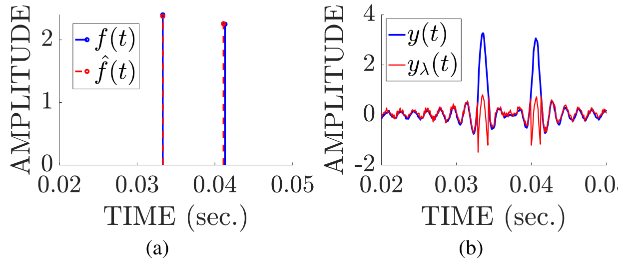


FIGURE 9. Reconstruction of FRI signal from the samples of modulo-hardware prototype demonstrated in [51]: (a) True (in blue) and reconstructed (in red) FRI signal for $L = 2$; (b) Lowpass filtered FRI signal (in blue) and its folded version (in red) with $\lambda = 1.25$. The FRI pulse $h(t)$ is a short pulse of bandwidth 30 kHz (Nyquist rate = 60 kHz). The folded signal is sampled six times below the Nyquist rate. The FRI parameters are nearly perfectly estimated from 10 samples by unfolding using first-order difference method and then by using the proposed annihilating filter approach.

non-idealities [51], [52], [53], [54]. In our work [51], we presented a modulo-ADC prototype that can operate at higher frequencies compared to the existing hardware prototypes and fold signals which are eight times larger than the ADC's dynamic range. We showed that the FRI signals consisting of a stream of Dirac impulses could be reconstructed perfectly from its lowpass, folded samples by using the unfolding algorithm presented in [16] and [17].

In this work, we consider lowpass filtered, modulo-folded samples from the hardware prototype and apply the first-order difference method for unfolding and the ASAF method for parameter estimation. The FRI signal and its reconstructed version are shown in Fig 9(a). The lowpass filtered signal, and its folded version are shown in Fig 9(a). We observe a near-perfect reconstruction of the FRI signal by sampling the signal at 10 kHz, which is six times below the Nyquist rate. For reconstruction, we used ten unfolded samples.

VIII. CONCLUSION

We consider the problem of sampling FRI signals under a modulo framework. We present theoretical guarantees and a practical algorithm for recovering FRI parameters from the modulo samples using an SoS kernel. Theoretical guarantees show that for unique recovery, one needs to sample above the RoI and consider as many samples as the number of unknowns of the FRI signal. The proposed practical algorithms operate at a lower sampling rate and require fewer samples than existing ones. The results enable the design of low-cost and high-dynamic-range ADCs with compactly supported kernels.

The unfolding algorithm is based on the first-order difference, which requires a high sampling rate compared to the RoI. As a future direction, we would like to use generalized FRI approaches to unfold and estimate FRI parameters jointly at lower sampling rates.

APPENDIX A
PROOF OF THEOREM 2

Proof: We first consider the necessary conditions and then prove sufficiency.

A. NECESSARY CONDITIONS

We show that for $K' \leq K$ there exist another trigonometric polynomial of order K whose modulo samples coincide with $y_\lambda(nT_s)$. We first consider the case when $K' < K$. Note that to uniquely determine $\{F(k\omega_0)\}_{k=-K}^{K}$ from the samples $y(nT_s)$ in (16) there should be more than or equal to $2K+1$ samples within a time interval of length T_d. To this end, the sampling interval should satisfy the inequality $T_s \leq \frac{2\pi}{T_d}$. Otherwise, there exists an alternative real-valued trigonometric polynomial

$$\hat{y}(t) = \sum_{k=-K}^{K} \hat{F}(k\omega_0) e^{\mathrm{j}k\omega_0 t} \tag{39}$$

such that $y(nT_s) = \hat{y}(nT_s)$. Hence, we have $y_\lambda(nT_s) = \hat{y}_\lambda(nT_s)$. This proves that for $K' < K$ one cannot uniquely recovery $y(nT_s)$ from $y_\lambda(nT_s)$.

For $K' = K$, although one can recover $\{F(k\omega_0)\}_{k=-K}^{K}$ from $y(nT_s)$, but we cannot uniquely recover $y(nT_s)$ from $y_\lambda(nT_s)$. To show this, we construct an alternative polynomial $\hat{y}(t)$ as

$$\hat{y}(t) = y(t) + 2p\lambda = \sum_{k=-K}^{K} \hat{F}(k\omega_0) e^{\mathrm{j}k\omega_0 t}, \tag{40}$$

where $p \in \mathbb{Z}$. The coefficients of the alternative polynomial are related to those of $y(t)$ as

$$\hat{F}(k\omega_0) = F(k\omega_0) + 2p\lambda\, \delta_K(k), \tag{41}$$

where δ_K is the Kronecker impulse. The samples of the alternative polynomial are given as $\hat{y}(nT_s) = y(nT_s) = 2p\lambda$ and hence we have that $\hat{y}_\lambda(nT_s) = y_\lambda(nT_s)$.

B. SUFFICIENT CONDITION

For $K' > K$ let there exist an alternate solution as in (39) Let us assume that there exists an alternate solution as in (39) such that $y_\lambda(nT_s) = \hat{y}_\lambda(nT_s)$. This implies that there exist samples $d(nT_s) \in \mathbb{Z}$ such that $y(nT_s) - \hat{y}(nT_s) = 2\lambda d(nT_s)$ or

$$\sum_{k=-K}^{K} (F(k\omega_0) - \hat{F}(k\omega_0)) e^{\mathrm{j}k\omega_0 nT_s} = \quad 2\lambda d(nT_s). \tag{42}$$

Since $T_s = \frac{T_d}{2K'+1}$ there are $2K'+1$ samples within an interval T_d. Without loss of generality let $n = 0, 1, \cdots, 2K'$ in (42). Then, by using the inverse discrete Fourier transform, we have

$$\sum_{n=0}^{2K'} d(nT_s) e^{-\mathrm{j}k\omega_0 nT_s} = \begin{cases} \dfrac{2K'+1}{2\lambda}(F(k\omega_0) - \hat{F}(k\omega_0)), & \\ \qquad \text{for} -K \leq k \leq K, & \\ 0, \quad \text{for} -K' \leq k < -K \text{ and } K < k \leq K'. & \end{cases} \tag{43}$$

This implies that the polynomial $D(z) = \sum_{n=0}^{2K'} d(nT_s) z^n$ has roots on the unit circle at $\{e^{-\mathrm{j}k\omega_0 T_s}\}_{-K' \leq k < -K}$ and $\{e^{-\mathrm{j}k\omega_0 T_s}\}_{K < k \leq K'}$. Note that the coefficients of the polynomials are integer-valued. Next, we use properties of integer valued polynomials [55, pp. 308-311] and show that $\{e^{-\mathrm{j}k\omega_0 T_s}\}_{k \in \{-K,\cdots,K\} \backslash \{0\}}$ too are zeros of $D(z)$. This implies $F(k\omega_0) = \hat{F}(k\omega_0)$, $k \in \{-K, \cdots, K\} \backslash \{0\}$ and uniqueness is established.

To this end, we first analyze the characteristics of the zeros of $D(z)$. In particular, consider the root $z_{K'} = e^{-\mathrm{j}K'\omega_0 T_s}$. By substituting $\omega_0 = \frac{2\pi}{T_d}$ and $T_s = \frac{T_s}{2K'+1}$, we have that $z_{K'} = e^{-\mathrm{j}K\omega_0 T_s} = e^{-\mathrm{j}\frac{K}{2K'+1}}$. This implies that $z_{K'}$is a $(2K'+1)$-th root of unity. Moreover, since $2K'+1$ is prime, $z_{K'}$ is $(2K'+1)$-th primitive root of unity. For any primitive root of unity, there exists a Cyclotomic polynomial $Q(z)$ such that $Q(z_{K'}) = 0$ where a Cyclotomic polynomial is a monic polynomial with integer coefficients. Importantly, it is the minimal polynomial over the field of rational numbers of any primitive nth-root of unity. Hence, the degree of $Q(z)$ is less than or equal to $D(z)$. From the polynomial remainder theorem, we have that

$$D(z) = A(z)Q(z) + R(z), \tag{44}$$

where $A(z)$ and $R(z)$ are polynomials with integer coefficients and the degree of R is less than that of Q. Since $D(z_{K'}) = Q(z_{K'}) = 0$ from (44) we have that $R(z_{K'}) = 0$. However, since $Q(z)$ is the minimal polynomial with integer coefficient with the root at $z_{K'}$, this implies that $R(z) = 0$. Since all the primitive roots of unity are zeros of the corresponding Cyclotomic polynomial, $Q(z)$ has zeros $\{e^{-\mathrm{j}\frac{k}{2K'+1}}\}_{k=-K'}^{-1}$. This implies that $F(k\omega_0) = \hat{F}(k\omega_0)$ for $k = -K, \cdots, -1, 1, \cdots, K$. In addition, we have that $F(0) \neq \hat{F}(0)$. In particular from (42) we conclude that $F(0) - \hat{F}(0) \in 2\lambda\mathbb{Z}$. This implies that the trigonometric polynomials $y(nT_s)$ and $\hat{y}(nT_s)$ are identical up to a constant factor which is a multiple of 2λ. ∎

APPENDIX B
NON-IDENTIFIABILITY WITH MISSING SAMPLE FOR $K = L$

Here we show that for $K = L$ one cannot uniquely identify $\{a_\ell, \tau_\ell\}_{\ell=1}^{L}$ from $\{\bar{s}[k]\}_{k=\mathcal{K}\backslash\{0\}}$ (cf. (21)) for $\beta \neq 0$. Specifically, there exist an alternative set of FRI parameters $\{\hat{a}_\ell, \hat{\tau}_\ell\}_{\ell=1}^{L}$ such that

$$\sum_{\ell=1}^{L} a_\ell\, e^{-\mathrm{j}k\omega_0 \tau_\ell} = \sum_{\ell=1}^{L} \hat{a}_\ell\, e^{-\mathrm{j}k\omega_0 \hat{\tau}_\ell}, \quad k = \mathcal{K}\backslash\{0\}, \tag{45}$$

where $K = L$. We show that (45) holds for a specific example. Let

$$\tau_\ell = \frac{T_d}{2L}\ell, \quad \text{and} \quad \hat{\tau}_\ell = \frac{T_d}{2L}(\ell + L), \quad \ell = 1, \cdots, L, \tag{46}$$

and

$$a_\ell = 1, \quad \text{and} \quad \hat{a}_\ell = -1, \quad \ell = 1, \cdots, L. \tag{47}$$

Then by substituting these time delays and amplitudes in

$$\sum_{\ell=1}^{L} a_\ell\, e^{-\mathrm{j}k\omega_0\tau_\ell} - \sum_{\ell=1}^{L} \hat{a}_\ell\, e^{-\mathrm{j}k\omega_0\hat{\tau}_\ell}, \tag{48}$$

we have

$$\sum_{\ell=1}^{L} a_\ell\, e^{-\mathrm{j}k\omega_0\tau_\ell} - \sum_{\ell=1}^{L} \hat{a}_\ell\, e^{-\mathrm{j}k\omega_0\hat{\tau}_\ell} = \sum_{\ell=1}^{2L} e^{-\mathrm{j}k\frac{2\pi}{2L}}. \tag{49}$$

The above right-hand-side sum is zero for $-L \leq k \leq L$ except for $k = 0$. This implies that (45) holds true for this set of FRI parameters. Hence, for $K = L$, identifiability is not achieved with ambiguity at $k = 0$.

APPENDIX C
PROOF OF LEMMA 1

Proof: Consider any eigenvalue $\gamma \in \sigma(\mathbf{S})$ with corresponding eigenvector $\mathbf{c} \in \mathcal{E}(\mathbf{S})$ such that $\mathbf{S}\,\mathbf{c} = \gamma\,\mathbf{c}$. By using the equality $\mathbf{S} = \bar{\mathbf{S}} - \bar{\beta}\mathbf{I}_{L+1}$ (cf. (17)), we have that $\bar{\mathbf{S}}\mathbf{c} = (\bar{\beta} + \gamma)\mathbf{c}$. This implies that if $\mathbf{c} \in \mathcal{E}(\mathbf{S})$ then $\mathbf{c} \in \mathcal{E}(\bar{\mathbf{S}})$. In addition, if $\gamma \in \sigma(\mathbf{S})$ then $\gamma + \bar{\beta} \in \sigma(\bar{\mathbf{S}})$.

By applying a similar approach, it can be verified that if $\bar{\mathbf{c}} \in \mathcal{E}(\bar{\mathbf{S}})$ then $\bar{\mathbf{c}} \in \mathcal{E}(\mathbf{S})$ and hence $\mathcal{E}(\bar{\mathbf{S}}) = \mathcal{E}(\mathbf{S})$. Further, if $\bar{\mathbf{S}}\bar{\mathbf{c}} = \bar{\gamma}\bar{\mathbf{c}}$ for any $\bar{\gamma} \in \sigma(\bar{\mathbf{S}})$ then $\mathbf{S}\bar{\mathbf{c}} = (\bar{\gamma} - \bar{\beta})\bar{\mathbf{c}}$ which implies that for every $\bar{\gamma} \in \sigma(\bar{\mathbf{S}})$ we have $\bar{\gamma} - \bar{\beta} \in \sigma(\mathbf{S})$ and hence $\sigma(\bar{\mathbf{S}}) = \bar{\beta} + \sigma(\mathbf{S})$. ■

APPENDIX D
UPPER BOUND ON *M*

We derive an upper bound on the number of level crossings of a trigonometric polynomial. Consider a K-th order trigonometric polynomial

$$y(t) = \sum_{k=-K}^{K} c_k\, F(k\omega_0) e^{-\mathrm{j}k\omega_0 t} \tag{50}$$

which is the same as the filtered FRI signal over an interval of length T_d. We are interested in finding the maximum number of times $y(t)$ crosses level sets $2\lambda\mathbb{Z}$. In the following, we denote the upper bound as M. To this end, let us consider the following lemma.

Lemma 2 (Level-Crossings of Trigonometric Polynomial): Consider a trigonometric polynomial $y(t)$ of order K and time period T_d as in (50). Consider an amplitude level l such that $|l| \leq \|y(t)\|_\infty$. Then $y(t)$ has a maximum of $2K$ crossings with level l within one time-period T_d.

Proof: The lemma is a direct consequence of the fact that a trigonometric polynomial of order K has a maximum $2K$ zeros within one time-period [56, p. 150]. Determining the level crossings is equivalent to solving for values of $t \in (0, T_d]$ such that $y(t) = l$ or, equivalently, $y(t) - l = 0$. Since $y(t) - l$ is another trigonometric polynomial of order K, it has a maximum of $2K$ zeros within one time period. ■

The result implies that if $2\lambda n_1 < \|y(t)\|_\infty$ for any integer n_1, then there will be at max $2K$ foldings corresponding to the level $2\lambda n_1$. Now the maximum number of level-sets $y(t)$ can cross is given by $\left\lfloor \frac{2\|y(t)\|_\infty}{\lambda} \right\rfloor$. Hence, M is upper-bounded as

$$M \leq \left\lfloor \frac{2\|y(t)\|_\infty}{\lambda} \right\rfloor 2K \tag{51}$$

The results indicate that the sampling rate and the number of samples required for perfect recovery, are inversely proportional to λ.

Using (50), we have that

$$|y(t)| \leq (2K+1) \max_{k\in\mathcal{K}} |c_k||F(k\omega_0)|. \tag{52}$$

From (3), we note that

$$|F(\omega)| \leq \|H(\omega)\|_\infty L a_{\max} = \|h\|_1 L a_{\max}. \tag{53}$$

Hence

$$\|y(t)\|_\infty = (2K+1)c_{\max}\|h\|_1 L a_{\max}, \tag{54}$$

where $c_{\max} = \max_{k\in\mathcal{K}} |c_k|$. From (54) and (51), the sampling rate and the number of samples required for uniquely identifying the FRI signal is on the order of K^2 where $K \geq L$.

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