Linear Precoding via Conic Optimization for Fixed MIMO Receivers

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Invited Paper

***Abstract*—In this paper, the problem of designing linear precoders for fixed multiple-input-multiple-output (MIMO) receivers is considered. Two different design criteria are considered. In the first, the transmitted power is minimized subject to signal-to-interference-plus-noise-ratio (SINR) constraints. In the second, the worst case SINR is maximized subject to a power constraint. It is shown that both problems can be solved using standard conic optimization packages. In addition, conditions are developed for the optimal precoder for both of these problems, and two simple fixed-point iterations are proposed to find the solutions that satisfy these conditions. The relation to the well-known uplink–downlink duality in the context of joint transmit beamforming and power control is also explored. The proposed precoder design is general, and as a special case, it solves the transmit rank-one beamforming problem. Simulation results in a multiuser system show that the resulting precoders can significantly outperform existing linear precoders.**

***Index Terms*—Linear precoding, multiple-input-multiple-output (MIMO) systems, optimization methods.**

I. Introduction

MULTIPLE-INPUT-MULTIPLE-OUTPUT (MIMO) systems arise in many modern communication channels, such as multiple-user communication [1], and multiple-antenna channels [2]. It is well known that the use of multiple antennas promises substantial capacity gains when compared with traditional single-antenna systems. In order to exploit these gains, the system must deal with the distortion caused by the channel and the interference. The conventional way to deal with these distortions is receiver optimization. Recently, the quest for better performance with lower complexity led researchers to also optimize the transmitter [3]–[11], and even to jointly optimize the transmitter and receiver [12]–[20]. This, as well as new results and algorithms in convex optimization theory [21], have significantly improved state of the art communication systems.

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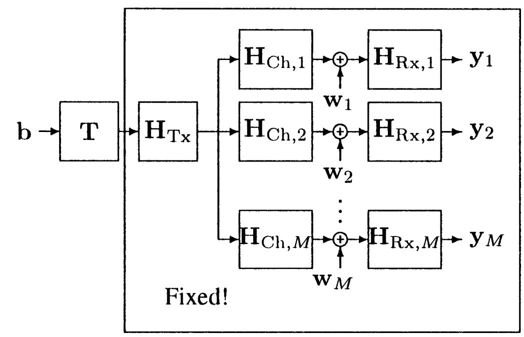

Fig. 1. Block diagram of a precoder for a fixed MIMO receiver.

In this paper, we explore the design of a centralized precoder given fixed linear MIMO transmitter, channel, and receiver (see Fig. 1). We define a precoder as a linear transformation on the transmitted symbols. If the precoded symbols are sent as is to the channel, then the precoder is the transmitter itself. However, in general, the precoded symbols may be transformed again before the channel. We refer to this transformation as the transmitter, and we assume that it is a fixed design parameter. The output of the transmitter is then sent over a fixed MIMO channel (or channels) and is received using a fixed linear receiver (or receivers).

There are many applications in which the transmitter and the receivers are fixed and the designer must resort to precoding. For example, consider the downlink channel of a multiuser system. In code-division multiplex-access (CDMA) systems, the transmitter is constrained to spreading using standardized signatures that cannot be altered. In addition, the receivers on the mobile hand sets are usually restricted to simple low computational complexity algorithms, e.g., matched filters (MFs), which are not necessarily optimal. Another example with growing interest is when the base station transmits using multiple antennas to multiple users using single-receive antennas. Each user has access only to its received signal and cannot cooperate with the other users. Thus, receive processing is practically impossible and the system must resort to precoding.

One of the first results on optimizing a precoder for a fixed linear MIMO model is due to [3] in the context of CDMA systems. Specifically, a precoder that applied a linear transformation on the transmitted symbols prior to the spreading was derived. This precoder inverted the channel at the transmitter side and is usually referred to as the transmit zero-forcing (ZF) precoder. The main drawback of the transmit ZF precoder is its degraded performance in low signal-to-noise-ratio (SNR) since inverting the channel increases the noise power. This motivated the design of transmit MF precoders and transmit rakes, which perform better in low SNR. In addition, transmit minimum mean-squared error (MMSE)[1] precoders that tried to compensate for the performance in the different SNR regions were derived [5]–[10].

Other precoders using different kinds of optimization criteria were also derived. Variants of the previous precoders were discussed in [22]–[27]. Linear precoders based on an approximate maximum-likelihood approach and maximum asymptotic multiuser efficiency with different power constraints were derived in [28]. A linear precoding technique based on a decomposition approach was proposed in [29], and a linear precoder design for nonlinear maximum-likelihood (ML) receivers was discussed in [30]. Among the nonlinear precoders are the Tomlinson Harashima precoder (THP) [31], the "Dirty Paper" precoder [32], and the vector perturbation precoder [9]. Another nonlinear precoder that optimizes the transmitted symbols vector itself was derived in [33].

The problem of precoder design is highly related to other problems in the literature. In this paper, we consider the design of linear precoders for fixed linear receivers. A related problem is the problem of jointly optimizing the precoder/transmitter and the receiver, which has been treated, e.g., in [12]–[20]. The design of the optimal signatures (which can be considered as a linear precoding scheme) for matched MMSE receivers was discussed in [34] and [35], whereas signature design for matched decision feedback receivers was explored in [36]. One of the interesting properties of these joint designs is that maximizing the signal-to-interference-plus-noise ratios (SINRs) is related to minimizing the MSE [16]. Thus, although different criteria have been explored, most of the research was dedicated to variants of the MMSE criterion.

Another related problem is the joint design of rank-one transmit beamforming design and optimal power control [37]–[39]. This problem is equivalent to precoding when there is no transmitter, i.e., the precoder itself is the transmitter. At first glance, it seems that the precoding problem can be solved by addressing the transmit beamforming problem and then compensating for the fixed transmitter. Unfortunately, this is not possible when the transmitter is rank deficient and cannot be inverted. In this aspect,[2] our problem is more general. Unlike the previous references regarding precoding, which usually dealt with the MSE criterion and its variants, the beamforming community has successfully managed to optimize SINR-based criteria, which are more related to practical performance measures, such as bit error rate (BER) and capacity. This problem is mathematically more difficult than MSE optimization. It was solved using an interesting duality between downlink and uplink beamforming [40], [41]. The uplink beamforming problem has been solved before in [42] and [43]. Using the duality, the downlink SINR problem can be handled as well [38], [44]. Recently, a nonlinear THP version of these papers was presented in [45].

[1]The name transmit MMSE is borrowed from the terminology of receiver design. These receivers do not minimize the mean-squared-error (MSE) between the received vector and the symbol vector. On the contrary, the transmit ZF minimizes it [3].

[2]On the other hand, the above references deal with beamforming for rank $r > 1$ and are therefore more general than precoding.

In the present paper, we integrate the ideas above in the context of MIMO precoding for fixed receivers. The design of most of the previous precoders is based on minimizing variants of the common MSE criterion. This criterion is usually computationally attractive and performs quite well. However, as far as the applications are concerned, the interesting and relevant criteria are BER and capacity, which are intimately associated with maximizing SINR [1]. Unlike joint optimization, optimizing the precoder to minimize MSE does not necessarily maximize SINR when the receiver is fixed. Thus, following the transmit beamforming approach, we focus on SINR-based criteria, and, in particular, try to optimize the worst SINR. We consider two design strategies: The first maximizes the worst SINR subject to an average power constraint, and the second minimizes the required average power subject to a constraint on the worst SINR. We prove that the proposed precoders have the attractive property of equal performance among all the subchannels.

Our precoder design is based on the powerful framework of convex optimization theory [21], which allows for efficient numerical solutions using standard optimization packages [46]. A brief review of such programs and their standard forms is provided in Section III. We then cast the precoder design problems as standard conic optimization packages. Specifically, we show that the power optimization problem can be formulated as a second-order cone program (SOCP) [47] or a semidefinite program (SDP) [48] [otherwise known as a linear matrix inequalities (LMI) program]. The SINR optimization can also be formulated as a standard conic program known as the generalized eigenvalue problem (GEVP) [49].

Next, we derive optimality conditions for both of the design problems by analyzing the Karush–Kuhn–Tucker (KKT) conditions for conic programs. We derive a simple expression for the structure of the optimal precoder as a function of the dual variables. The conditions can be used to verify whether a proposed solution is optimal. For example, using these conditions, it is easy to show that the MMSE precoder proposed in [5]–[7] does not necessarily maximize the worst SINR, except in the case of a symmetric channel. Another use for these conditions is as a stopping criteria in previous iterative optimization algorithms.

Probably the most important use of the optimality conditions is in deriving new design algorithms. Using the conditions, we provide a simple fixed-point iteration that is guaranteed to converge to the solution of the power optimization. As a special case, this simple iteration can solve the well-known rank-one beamforming problem. This allows a simple solution to the problem without the need for special optimization packages. A similar fixed-point iteration is derived for the SINR optimization problem without a convergence proof. In comparison to the downlink–uplink duality-based solutions, our simple fixed-point iterations are considerably more appealing. In addition, following [39], we derive an alternative approach for satisfying the optimality conditions in the power optimization through a dual SDP/LMI program.

One of the advantages of our proposed algorithms is their robustness to the rank of the effective channels. Most of the previous precoders assume a full-rank effective channel. For example, one cannot decorrelate the channel in [3] if the channel is rank deficient, as is the case when the number of users is greater than the spreading factor (or the number of transmit antennas). Our design algorithms, both the conic solutions and the fixed-point iterations, are indifferent to the rank of the channel, and are therefore applicable to such scenarios as well. In addition, following [34] and [35], which addressed this problem in the context of optimal sequences design for MMSE receivers, we provide an upper bound for the maximal feasible SINRs in these cases.

An interesting result of our precoders is their performance in symmetric systems. In this case, our precoders admit simple closed-form expressions that have already been derived in [5]–[9] through different considerations and for general channels. Using our optimality conditions, it is easy to show that these precoders maximize the worst SINR in symmetric channels. A realistic example of such systems is a CDMA scheme, using pseudonoise (PN) sequences as signatures. We analytically show that the achievable SINRs using these precoders with MF receivers is identical to those obtained by using MMSE receivers with no precoders. This result is interesting, as it allows for each user to use a simple receiver that does not require neither the knowledge of all the other signatures nor a matrix inversion. It is important to note that this feature does not extend to nonsymmetric channels.

The paper is organized as follows. We begin in Section II by introducing the problem formulation. A brief review of conic optimization is provided in Section III. The power optimization problem is explored in Section IV, in which we discuss its feasibility and provide standard conic optimization solutions. In order to improve our design algorithms and in order to gain more insight into the problem, we then provide optimality conditions and suggest a simple fixed-point iteration for finding the variables that satisfy them. Next, in Section V, we follow the same steps for the SINR optimization problem. A few special cases for which a closed-form solution exists are explored in Section VI. In Section VII, we illustrate the use of the aforementioned precoders in the context of multiple-user communication systems.

The following notation is used. Boldface upper-case letters denote matrices, boldface lower-case letters denote column vectors, and standard lower-case letters denote scalars. The superscripts $(\cdot)^T$, $(\cdot)^*$, $(\cdot)^H$, $(\cdot)^{-1}$, and $(\cdot)^\dagger$ denote the transpose, the complex conjugate, the Hermitian, the matrix inverse operators, and the Moore Penrose pseudoinverse, respectively. $[\mathbf{X}]_{i,j}$ denotes the (ith, jth) element of the matrix $\mathbf{X}$. By $\mathrm{diag}\{x_i\}$, we denote a diagonal matrix with x_i being the (ith, ith) element; by $\mathrm{vec}(\mathbf{X})$, we denote stacking the elements of $\mathbf{X}$ in one long column vector; by $\mathbf{e}_i$, we denote a zeros vector with a one at the ith element; by $\mathbf{1}$, we denote an all ones vector; and by $\mathbf{I}$, we denote the identity matrix of appropriate size. $\mathrm{Tr}\{\cdot\}$, $\Re\{\cdot\}$, $|\cdot|$, $\|\cdot\|$, and $|||\cdot|||_\infty$ denote the trace operator, the real part, the absolute value, the standard Euclidean norm, and the induced row sum matrix norm, respectively. Finally, $\mathbf{X} \succeq 0$ denotes that the matrix $\mathbf{X}$ is a Hermitian positive semidefinite matrix, and $\mathcal{N}\{\cdot\}$ denotes the Null space operator.

II. Problem Formulation

Consider a general, block oriented, MIMO communication system with a centralized transmitter. At each time instant, a block of symbols is modulated and transmitted over the channels. The possibly distorted output is then processed at the receivers in a linear fashion, as depicted in Fig. 1. Denoting by $\mathbf{y}_i$ the length L output of the mth receiver, for $m = 1, \ldots, M$, we have that

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_M \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{\mathrm{Rx},1}\mathbf{H}_{\mathrm{Ch},1} \\ \vdots \\ \mathbf{H}_{\mathrm{Rx,M}}\mathbf{H}_{\mathrm{Ch},M} \end{bmatrix} \mathbf{H}_{\mathrm{Tx}}\mathbf{b} + \begin{bmatrix} \mathbf{H}_{\mathrm{Rx},1}\mathbf{w}_1 \\ \vdots \\ \mathbf{H}_{\mathrm{Rx},M}\mathbf{w}_M \end{bmatrix} \tag{1}$$

where the matrices $\mathbf{H}_{\mathrm{Rx},m}$ and $\mathbf{H}_{\mathrm{Ch},m}$ denote the receiver and channel associated with the mth user, the matrix $\mathbf{H}_{\mathrm{Tx}}$ is the centralized transmitter, $\mathbf{b}$ is the length $K = M \cdot L$ vector of independent, and unit variance transmitted symbols, and $\mathbf{w}_i$ are the noise vectors. The noise vectors may be correlated, and the channels are completely arbitrary. The only restriction is that the transmitter is centralized and has access to all of the K transmit components.

Our problem formulation assumes a single stream per MIMO dimension, i.e., the length of $\mathbf{b}$ is equal to the length of $\mathbf{y}$. However, these streams can be dedicated to a single user or to multiple users. Three specific examples for which this model holds are given below.

- *Point-to-point multiple antenna system*—A single user, point-to-point communication system using L multiple receive and L transmit antennas is a special case of (1) with $M = 1$.
- *CDMA system*—The downlink channel of a CDMA system with L users is a special case of (1) with $L = 1$, where $\mathbf{H}_{\mathrm{Tx}}$ is a signature matrix whose columns are the signatures of each of the users, $\mathbf{H}_{\mathrm{Ch},i} = \mathbf{I}$ and $\mathbf{H}_{\mathrm{Rx},i}$ are row vectors representing the linear receive filters of each of the users.
- *Transmit beamforming*—A multiuser system in which L transmit antennas signal to L users each using a single receive antenna is a special case of (1) with $L = 1$. Here, $\mathbf{H}_{\mathrm{Tx}}$ is a beamforming matrix whose columns are the antenna weights of each of the L users, $\mathbf{H}_{\mathrm{Ch},i}$ are row vectors that represent the paths from the transmit antennas to the ith receive antenna, and $\mathbf{H}_{\mathrm{Rx},i}$ are arbitrary scalars.

In the sequel, we will assume that the transmitter $\mathbf{H}_{\mathrm{Tx}}$, the channels $\mathbf{H}_{\mathrm{Ch},i}$ and the receivers $\mathbf{H}_{\mathrm{Rx},i}$ are fixed and cannot be altered due to budget restrictions, standardization, or physical problems. Given this fixed structure, we will try to improve the performance by introducing a linear precoder. The precoder, denoted by $\mathbf{T}$, linearly transforms the original symbol vector prior to the transmission so that the outputs of the receiver are now given by

$$\underbrace{\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_M \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} \mathbf{H}_{\mathrm{Rx},1}\mathbf{H}_{\mathrm{Ch},1} \\ \vdots \\ \mathbf{H}_{\mathrm{Rx},M}\mathbf{H}_{\mathrm{Ch},M} \end{bmatrix}}_{\mathbf{H}_{\mathrm{RxCh}}} \mathbf{H}_{\mathrm{Tx}}\mathbf{T}\mathbf{b} + \underbrace{\begin{bmatrix} \mathbf{H}_{\mathrm{Rx},1}\mathbf{w}_1 \\ \vdots \\ \mathbf{H}_{\mathrm{Rx},M}\mathbf{w}_M \end{bmatrix}}_{\mathbf{w}}. \tag{2}$$

For ease of representation, we will use the following notation:

$$\mathbf{y} = \mathbf{H}\mathbf{T}\mathbf{b} + \mathbf{w} \tag{3}$$

where $\mathbf{H} = \mathbf{H}_{\mathrm{RxCh}}\mathbf{H}_{\mathrm{Tx}}$ and the rest of the variables are defined in (2).

Our goal is to improve the system performance by optimally designing the precoder. The system performance is usually quantified by its quality of service (QoS) and the resources it uses. The most common QoS metrics are BER and capacity, both of which are highly related to the output SINRs, and in particular to the worst SINR. In our model, the output SINR of the ith subchannel is defined as

$$\mathrm{SINR}_i = \frac{|[\mathbf{HT}]_{i,i}|^2}{\sum_{j\neq i}|[\mathbf{HT}]_{i,j}|^2 + \sigma_i^2} \tag{4}$$

for $i = 1,\ldots,K$, where $\sigma_i^2 = E\{|w_i|^2\} > 0$. Another range of criteria deal with the use of system resources, e.g., peak-to-average ratio, or maximal transmitted power. The most common resource measure is average transmitted power, which is defined as

$$P = E\{\|\mathbf{H}_{\mathrm{Tx}}\mathbf{Tb}\|^2\} = \mathrm{Tr}\left\{\mathbf{T}^H\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}\mathbf{T}\right\}. \tag{5}$$

It is easy to see that the SINR metric and average power metric conflict. One cannot maximize the SINRs while also minimizing the power, and vice versa. Depending on the application, the designer must decide which criteria is stricter. We therefore consider one of the following two complementary strategies. The first optimization strategy seeks to minimize the average transmitted power subject to QoS constraints. This criterion is interesting from a system-level perspective. Given the required QoS, the system tries to satisfy it with minimum transmitted power [17], [38], as follows:

$$\boldsymbol{\mathcal{P}}(\gamma_o) = \begin{cases} \min_{\mathbf{T}} & \mathrm{Tr}\left\{\mathbf{T}^H\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}\mathbf{T}\right\} \\ \text{s.t.} & \min_i \frac{|[\mathbf{HT}]_{i,i}|^2}{\sum_{j\neq i}|[\mathbf{HT}]_{i,j}|^2+\sigma_i^2} \geq \gamma_o \end{cases} \tag{6}$$

where $\gamma_o > 0$ is the given worst SINR constraint.

The second strategy is maximizing the minimal SINR subject to a power constraint [16], [44]. This problem formulation is interesting when the power constraint is a strict system restriction that cannot be relaxed. In this case, the problem can be formulated as

$$\boldsymbol{\mathcal{S}}(P_o) = \begin{cases} \max_{\mathbf{T}} & \min_i \frac{|[\mathbf{HT}]_{i,i}|^2}{\sum_{j\neq i}|[\mathbf{HT}]_{i,j}|^2+\sigma_i^2} \\ \text{s.t.} & \mathrm{Tr}\left\{\mathbf{T}^H\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}\mathbf{T}\right\} \leq P_o \end{cases} \tag{7}$$

where $P_o > 0$ is the given power constraint.

Note that although we are optimizing the minimum SINR in both problems, it is easy to see that at the optimal solution of both problems all users will attain equal SINRs (see also [38]). In other words, the above design criteria both promise fairness among all the substreams. This is an important property in MIMO communication systems. In systems where some streams demand different QoS, e.g., systems with voice and data streams, the designer can replace each SINR_i in the optimizations with SINR_i/ρ_i, where ρ_i are constant weights that denote the importance of the substreams. This will ensure weighted fairness among the streams.

One of the main observations of our work is that both optimization problems (6) and (7) can be solved using standard conic optimization algorithms. Therefore, in the next section, we review these algorithms.

III. Review of Conic Optimization

In recent years, there has been considerable progress and development of efficient algorithms for solving a variety of optimization problems. In order to use these algorithms, one must reformulate the problem into a standard form that the algorithms are capable of dealing with. In this section, we will briefly review the three formulations that we use in the paper: SOCP, SDP, and GEVP programming.

The most widely researched field in optimization is convex optimization. A convex program is a program with a convex objective function and convex constraints. It is well known that in such programs a local minimum is also a global minimum. The most common convex program is probably the linear program (LP) [21], i.e., an optimization with a linear objection function and linear (affine) constraints. Recent advances in convex optimization generalize the results and algorithms of LPs to more complicated convex programs. Special attention is given to conic programs, i.e., LPs with generalized inequalities. The two standard conic programs are SOCP and SDP optimization. The standard form of an SOCP is [47]

$$\mathrm{SOCP}: \begin{cases} \min_{\mathbf{x}} & \Re\{\mathbf{f}^H\mathbf{x}\} \\ \text{s.t.} & \begin{bmatrix} \mathbf{c}_i^H\mathbf{x} + d_i \\ \mathbf{A}_i^H\mathbf{x} + \mathbf{b}_i \end{bmatrix} \succeq_K 0, \quad i = 1,\ldots,N \end{cases} \tag{8}$$

where the optimization variable is the vector $\mathbf{x}$ of length n and $\mathbf{f}, \mathbf{A}_i, \mathbf{b}_i, \mathbf{c}_i$, and d_i for $i = 1,\ldots,N$ are the data parameters of appropriate sizes. The notation $\succeq_K$ denotes the following generalized inequality:

$$\begin{bmatrix} z \\ \mathbf{z} \end{bmatrix} \succeq_K 0 \Leftrightarrow \|\mathbf{z}\| \leq z. \tag{9}$$

The standard form of an SDP is [48]

$$\mathrm{SDP}: \begin{cases} \min_{\mathbf{x}} & \Re\{\mathbf{f}^H\mathbf{x}\} \\ \text{s.t.} & \mathbf{A}(\mathbf{x}) \succeq 0 \end{cases} \tag{10}$$

where $\mathbf{A}(\mathbf{x}) = \mathbf{A}_0 + \sum_{i=1}^n x_i\mathbf{A}_i$ is a Hermitian matrix that depends affinely on $\mathbf{x}$. The data parameters are the Hermitian matrices $\mathbf{A}_i$ for $i = 0,\ldots,n$. The notation $\succeq$ denotes the positive semidefinite generalized inequality. A simple case of an SDP is an SOCP. For example, each of SOC constraints in (8) can be written as an LMI [21], as follows:

$$\begin{bmatrix} \mathbf{c}_i^H\mathbf{x} + d_i & \mathbf{x}^H\mathbf{A}_i + \mathbf{b}_i^H \\ \mathbf{A}_i^H\mathbf{x} + \mathbf{b}_i & \left(\mathbf{c}_i^H\mathbf{x} + d_i\right)\mathbf{I} \end{bmatrix} \succeq 0. \tag{11}$$

A common optimization package designed to solve SOCP and SDP is *SEDUMI* [46].

Although most of the research in the field of optimization concerns convex programs, due to their importance, some cases of nonconvex problems have also been investigated. Among them is the GEVP [49], which is not convex but can still be efficiently solved. Its standard form is

$$\text{GEVP}: \begin{cases} \min_{\beta,\mathbf{x}} & \beta \\ \text{s.t.} & \beta\mathbf{B}(\mathbf{x}) - \mathbf{A}(\mathbf{x}) \succeq 0 \\ & \mathbf{B}(\mathbf{x}) \succeq 0 \\ & \mathbf{C}(\mathbf{x}) \succeq 0 \end{cases} \tag{12}$$

where β is a real-valued optimization variable and $\mathbf{A}(\mathbf{x}) = \mathbf{A}_0 + \sum_{i=1}^n x_i\mathbf{A}_i, \mathbf{B}(\mathbf{x}) = \mathbf{B}_0 + \sum_{i=1}^n x_i\mathbf{B}_i$ and $\mathbf{C}(\mathbf{x}) = \mathbf{C}_0 + \sum_{i=1}^n x_i\mathbf{C}_i$ are Hermitian matrices that depend affinely on $\mathbf{x}$. The data parameters are the Hermitian matrices $\mathbf{A}_i, \mathbf{B}_i$ and $\mathbf{C}_i$ for $i = 0, \ldots, n$. The name of the GEVP arises from its resemblance to the well-known problem of minimizing the maximal generalized eigenvalue of the pencil $[\mathbf{A}, \mathbf{B}]$, i.e., minimizing the largest β such that $\mathbf{A}\mathbf{v} = \beta\mathbf{B}\mathbf{v}$. It is easy to show that this problem can be expressed as

$$\begin{cases} \min_{\beta} & \beta \\ \text{s.t.} & \beta\mathbf{B} - \mathbf{A} \succeq 0 \end{cases} \tag{13}$$

which is, of course, a simple SDP. The GEVP generalizes this program to the case where $\mathbf{A}$ and $\mathbf{B}$ also depend on the optimization variables.

IV. Power Optimization

In this section, we consider the power optimization subject to SINR constraints, i.e., the $\mathcal{P}$ problem of (6). We begin in Section IV-A by discussing its feasibility and then provide a few alternative approaches for its solution. In particular, in Section IV-B, we derive a solution to the problem that is based on standard SOCP or SDP optimization packages. Next, in Section IV-C, we develop optimality conditions for this problem and use them to derive two alternative solutions. For completeness, in Section IV-D, we discuss the uplink–downlink duality in the context of the power optimization.

A. Feasibility

The first important property of any optimization problem is its feasibility (admissibility), i.e., whether a solution exists. In other words, we need to verify whether for a given γ_o there exists a $\mathbf{T}$ such that

$$\min_i \frac{|[\mathbf{HT}]_{i,i}|^2}{\sum_{j\neq i} |[\mathbf{HT}]_{i,j}|^2 + \sigma_i^2} \geq \gamma_o. \tag{14}$$

Since we have assumed that the noise variances are positive, the SINRs are strictly lower than the signal-to-interference ratios (SIRs), as follows:

$$\frac{|[\mathbf{HT}]_{i,i}|^2}{\sum_{j\neq i} |[\mathbf{HT}]_{i,j}|^2 + \sigma_i^2} < \frac{|[\mathbf{HT}]_{i,i}|^2}{\sum_{j\neq i} |[\mathbf{HT}]_{i,j}|^2} \tag{15}$$

for $i = 1, \ldots, K$. By scaling $\mathbf{T}$ to $a\mathbf{T}$ for large enough $a > 0$, the difference between the SIRs and the SINRs can be made insignificant. Therefore, for the sake of examining the feasibility, the interesting metrics are the SIRs. A condition for feasibility is provided in the following proposition.

Proposition 1: There exists a $\mathbf{T}$ such that

$$\min_i \frac{|[\mathbf{HT}]_{i,i}|^2}{\sum_{j\neq i} |[\mathbf{HT}]_{i,j}|^2} \geq \gamma_o \tag{16}$$

only if

$$\gamma_o \leq \frac{1}{\frac{K}{\text{rank}(\mathbf{H})} - 1}. \tag{17}$$

Proof: In order to prove the proposition, we must upper bound the minimal SIR, as follows:

$$\min_i \frac{|[\mathbf{HT}]_{i,i}|^2}{\sum_{j\neq i} |[\mathbf{HT}]_{i,j}|^2} = \min_i \frac{1}{\frac{1}{\xi_i} - 1} = \frac{1}{\frac{1}{\min_i \xi_i} - 1} \tag{18}$$

where $\xi_i = (|[\mathbf{HT}]_{i,i}|^2)/([\mathbf{HTT}^H\mathbf{H}^H]_{i,i})$, and we have used the monotonicity of $f(a) = 1/(1/a - 1)$ in $a < 1$. Due to monotonicity, we can bound $f(a)$ by bounding its argument. Thus, we now develop a bound on the minimum ξ_i. Let $\mathbf{HT}$ have a singular value decomposition (SVD) $\mathbf{HT} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^H$, where $\mathbf{U}$ and $\mathbf{V}$ are semi-unitary $K \times r$ matrices, $\boldsymbol{\Lambda}$ is an $r \times r$ diagonal matrix, and $r = \text{rank}(\mathbf{HT})$. Then

$$\xi_i = \frac{\left|\mathbf{u}_i^H\boldsymbol{\Lambda}\mathbf{v}_i\right|^2}{\mathbf{u}_i^H\boldsymbol{\Lambda}^2\mathbf{u}_i}, \quad i = 1, \ldots, K \tag{19}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ are the ith columns of $\mathbf{U}^H$ and $\mathbf{V}^H$, respectively. For every $i = 1, \ldots, K$, we can bound (19) by applying the Cauchy–Schwarz inequality to the vectors $\boldsymbol{\Lambda}\mathbf{u}_i$ and $\mathbf{v}_i$

$$\left|\mathbf{u}_i^H\boldsymbol{\Lambda}\mathbf{v}_i\right|^2 \leq \left(\mathbf{v}_i^H\mathbf{v}_i\right)\left(\mathbf{u}_i^H\boldsymbol{\Lambda}^2\mathbf{u}_i\right), \quad i = 1, \ldots, K. \tag{20}$$

Since $\mathbf{v}_i^H\mathbf{v}_i = [(\mathbf{HT})^\dagger\mathbf{HT}]_{i,i}$, we conclude that

$$\xi_i \leq [(\mathbf{HT})^\dagger\mathbf{HT}]_{i,i}, \quad i = 1, \ldots, K. \tag{21}$$

Thus, the minimum ξ_i is bounded by

$$\begin{aligned} \min_i \xi_i &\leq \frac{1}{K}\sum_{i=1}^K \xi_i \leq \frac{1}{K}\sum_{i=1}^K [(\mathbf{HT})^\dagger\mathbf{HT}]_{i,i} \\ &= \frac{1}{K}\text{Tr}\{(\mathbf{HT})^\dagger\mathbf{HT}\} = \frac{\text{rank}(\mathbf{HT})}{K} \leq \frac{\text{rank}(\mathbf{H})}{K}. \end{aligned} \tag{22}$$

Substituting (22) into (18) yields the required result. ■

If the effective channel $\mathbf{H}$ is full rank, then the condition results in $\gamma_o \leq \infty$, i.e., any SIR is feasible. This is easily verified as the condition in (14) can be satisfied by choosing $\mathbf{T} = a\mathbf{H}^{-1}$ for large enough $a > 0$. This choice of precoder inverts the channel and eliminates all interference.

Unfortunately, when the effective channel is rank deficient, the interference cannot be eliminated, and there is an upper bound on the maximal SIRs. Similar conditions were provided in [34] in the context of optimal signature design using MMSE receivers (which is a special case of a MIMO system) and, in [17], in the context of joint transmit and receive processing. In this literature, it was shown that the condition of Proposition 1 is necessary and sufficient for feasibility using MMSE receivers. In our case, the receivers are fixed, and therefore the condition is only necessary. In general, we cannot always attain the bound when the receiver is fixed. Two simple examples for channels in which the bound cannot be achieved are a diagonal $\mathbf{H}$ with $K-\operatorname{rank}(\mathbf{H})$ diagonal zeros, or a channel $\mathbf{H}$ with two identical rows. In both of these examples, it is easy to see that, no matter what the precoder is, we will not attain the bound.

Nonetheless, experimenting with arbitrary channels shows that in almost all practical channels the bound can be achieved even for a fixed suboptimal receiver. For example, consider a rank $K-1$ channel $\mathbf{H}$ with the normalized null vector $\mathbf{u} \in \mathcal{N}\{\mathbf{H}^H\}$. Except for the case in which $u_i = 0$ for some $i = 1,\ldots,K$, the bound can always be attained by choosing

$$\mathbf{T} = \mathbf{H}^{\dagger}\operatorname{diag}\{1/u_i^*\}\mathbf{Q} \tag{23}$$

where $\mathbf{Q}$ is a matrix with unit diagonal elements and $[\mathbf{Q}]_{i,j} = -1/(K-1)$ for the nondiagonal $i \neq j$ elements. This is easily shown by considering the following chain:

$$\mathbf{HT} = \mathbf{HH}^{\dagger}\operatorname{diag}\{1/u_i^*\}\mathbf{Q} = \operatorname{diag}\{1/u_i^*\}\mathbf{Q} \tag{24}$$

where we have used $\mathbf{HH}^{\dagger} = \mathbf{I} - \mathbf{uu}^H$ and the fact that $\mathbf{1} \in \mathcal{N}\{\mathbf{Q}\}$. Substituting the above $\mathbf{HT}$ into the SIRs yields the maximal SIRs in rank $K-1$ channels, as follows:

$$\frac{|[\mathbf{HT}]_{i,i}|^2}{\sum_{j\neq i}|[\mathbf{HT}]_{i,j}|^2} = \frac{1}{\frac{K}{K-1}-1}, \quad i = 1,\ldots,K. \tag{25}$$

B. Conic Optimization Solution

We now show that the $\boldsymbol{\mathcal{P}}$ problem of (6) can be represented as a standard conic optimization program. Thus, using off-the-shelf optimization packages, we can numerically verify its feasibility and find its optimal solution. In order to use the standard forms of the conic programs, we must cast our problem constraints using the standard notations described in Section III.

Using a real-valued slack variable P_o, the program can be rewritten as

$$\boldsymbol{\mathcal{P}}(\gamma_o): \begin{cases} \min_{\mathbf{T},P_o} & P_o \\ \text{s.t.} & \frac{|[\mathbf{HT}]_{i,i}|^2}{\sum_{j\neq i}|[\mathbf{HT}]_{i,j}|^2+\sigma_i^2} \geq \gamma_o, \\ & \qquad i=1,\ldots,K \\ & \operatorname{Tr}\left\{\mathbf{T}^H\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}\mathbf{T}\right\} \leq P_o. \end{cases} \tag{26}$$

The argument $\mathbf{T}$ of the $\boldsymbol{\mathcal{P}}$ program is defined up to a diagonal phase scaling on the right, i.e., if $\mathbf{T}$ is optimal, then $\mathbf{T}\operatorname{diag}\{e^{j\phi_i}\}$, where ϕ_i for $i=1,\ldots,K$ are arbitrary phases, is also optimal. This is easy to verify, as the phases do not change the objective nor the constraints. Therefore, we can restrict ourselves to precoders in which $[\mathbf{HT}]_{i,i} \geq 0$ for $i=1,\ldots,K$, i.e., each has a nonnegative real part and a zero imaginary part. Taking this into account, we now recast the SINR constraints in standard form. Rearranging the constraints and using matrix notations, the constraints yield

$$\left(1+\frac{1}{\gamma_o}\right)|[\mathbf{HT}]_{i,i}|^2 \geq \left\| \begin{matrix}\mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i\end{matrix} \right\|^2, \quad i=1,\ldots,K. \tag{27}$$

Since $[\mathbf{HT}]_{i,i} \geq 0$ for $i=1,\ldots,K$, we can take the square root of $|[\mathbf{HT}]_{i,i}|^2$, resulting in

$$\sqrt{1+\frac{1}{\gamma_o}}\,[\mathbf{HT}]_{i,i} \geq \left\| \begin{matrix}\mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i\end{matrix} \right\|, \quad i=1,\ldots,K \tag{28}$$

which can be written as the SOCs

$$\begin{bmatrix} \sqrt{1+\frac{1}{\gamma_o}}[\mathbf{HT}]_{i,i} \\ \mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i \end{bmatrix} \succeq_K 0, \quad i=1,\ldots,K. \tag{29}$$

Similarly, the power constraint in (26) can be reformulated using the $\operatorname{vec}(\cdot)$ operator as $\|\operatorname{vec}(\mathbf{H}_{\mathrm{Tx}}\mathbf{T})\| \leq \sqrt{P_o}$, which is equivalent to the SOC

$$\begin{bmatrix} \sqrt{P_o} \\ \operatorname{vec}(\mathbf{H}_{\mathrm{Tx}}\mathbf{T}) \end{bmatrix} \succeq_K 0. \tag{30}$$

Using (29) and (30), and denoting $p = \sqrt{P_o}$, the program (6) can be cast in the standard SOCP form [47], as follows:

$$\boldsymbol{\mathcal{P}}(\gamma_o): \begin{cases} \min_{\mathbf{T},p} & p \\ \text{s.t.} & \begin{bmatrix} \sqrt{1+\frac{1}{\gamma_o}}[\mathbf{HT}]_{i,i} \\ \mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i \end{bmatrix} \succeq_K 0, \\ & \qquad i=1,\ldots,K \\ & \begin{bmatrix} p \\ \operatorname{vec}(\mathbf{H}_{\mathrm{Tx}}\mathbf{T}) \end{bmatrix} \succeq_K 0. \end{cases} \tag{31}$$

Thus, it can be efficiently solved using any standard SOCP package [46]. Such a solver can also numerically determine the feasibility of the optimization problem. A similar approach was taken in [39] in the context of transmit beamforming.

As explained in Section III, each SOC constraint can be replaced with an SDP constraint using (11). Thus, the problem can also be expressed as a standard SDP, as follows:

$$\boldsymbol{\mathcal{P}}(\gamma_o): \begin{cases} \min_{\mathbf{T},p} & p \\ \text{s.t.} & \mathbf{A}_i(\mathbf{T}) \succeq 0, \quad i=1,\ldots,K \\ & \mathbf{C}(\mathbf{T}) \succeq 0 \end{cases} \tag{32}$$

where

$$\mathbf{A}_i(\mathbf{T}) = \begin{bmatrix} \sqrt{1+\frac{1}{\gamma_o}}[\mathbf{HT}]_{i,i} & \begin{bmatrix}\mathbf{e}_i^H\mathbf{HT} & \sigma_i\end{bmatrix} \\ \begin{bmatrix}\mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i\end{bmatrix} & \sqrt{1+\frac{1}{\gamma_o}}[\mathbf{HT}]_{i,i}\mathbf{I} \end{bmatrix} \tag{33}$$

for $i=1,\ldots,K$, and

$$\mathbf{C}(\mathbf{T}) = \begin{bmatrix} p & \operatorname{vec}^H(\mathbf{H}_{\mathrm{Tx}}\mathbf{T}) \\ \operatorname{vec}(\mathbf{H}_{\mathrm{Tx}}\mathbf{T}) & p\mathbf{I} \end{bmatrix}. \tag{34}$$

However, solving SOCPs via SDP is not very efficient. Interior point methods that solve SOCP directly have a much better worst case complexity than their SDP counterparts [47].

It is important to note that the above formulations are general and do not depend on the rank of the channel. Thus, these solutions are also appropriate for rank-deficient channels.

C. Optimality Conditions

In this section, we will derive the KKT optimality conditions for the power optimization. These conditions provide more insight into the solution. In particular, we derive a simple structure for the optimal solution based on the Lagrange dual variables. Given this structure, we propose two alternative methods for finding the dual variables. In Section IV-C-1), we derive a simple fixed-point iteration that converges to these variables. The computational complexity of this approach is lower than that of the conic solution. Moreover, this solution does not require any external conic package, which is not always available. Alternatively, in Section IV-C-2), we propose a dual SDP program, whose optimal arguments are the necessary variables. The main results are summarized in the following theorem.

Theorem 1: Consider the power optimization program $\mathcal{P}$ of (6). Define the dual variables $\lambda_i > 0$ for $i = 1 \ldots K$ and denote $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_i\}$ and $\mathbf{G}(\lambda_i) = \mathbf{H}^H\mathbf{\Lambda}\mathbf{H} + \mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}$. If there exist $\lambda_i > 0$ such that

$$\gamma_o = \frac{1}{\frac{1}{[\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{H}\mathbf{G}^{\dagger}(\lambda_i)\mathbf{H}^H\mathbf{\Lambda}^{\frac{1}{2}}]_{i,i}} - 1}, \quad i = 1, \ldots, K \tag{35}$$

holds, then the program is strictly feasible. Moreover, if the condition in (35) holds, then the optimal $\mathbf{T}$ is of the form

$$\mathbf{T} = \mathbf{G}^{\dagger}(\lambda_i)\mathbf{H}^H\mathbf{\Lambda}^{\frac{1}{2}}\mathrm{diag}\{\delta_i\} \tag{36}$$

where δ_i are the positive weights that allocate the power between the users, as follows:

$$\delta_i = \sqrt{\sum_j \left[\left(\frac{\gamma_o}{1+\gamma_o}\mathbf{I} - \mathbf{F}\right)^{-1}\right]_{i,j} \lambda_j\sigma_j^2} \tag{37}$$

$$[\mathbf{F}]_{i,j} = \left[\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{H}\mathbf{G}^{\dagger}(\lambda_i)\mathbf{H}^H\mathbf{\Lambda}^{\frac{1}{2}}\right]_{i,j}^2 \tag{38}$$

for $i, j = 1, \ldots, K$. This structure of $\mathbf{T}$ is unique within the range of $\mathbf{H}_{\mathrm{Tx}}^H$. At this optimal solution, all the constraints are active, i.e., there are equal SINRs for all the subchannels. The optimal objective value is

$$P_o = \sum_i \lambda_i\sigma_i^2. \tag{39}$$

Proof: The proof consists of two parts. First, we show that if (35) holds, then the problem is strictly feasible. Next, assuming it is strictly feasible, we will use the KKT optimality conditions to show that the proposed solution is necessary and sufficient.

We begin by proving that if (35) holds, then the proposed solution in (36)–(38) is feasible. First, let us prove that this solution exists, i.e., that the matrix $[\gamma_o/(1+\gamma_o)\mathbf{I} - \mathbf{F}]$ in (37) is invertible and that the argument of the squared root is nonnegative. The matrix is invertible since the maximal eigenvalue of $\mathbf{F}$ is less than $\gamma_o/(1+\gamma_o)$, as follows:

$$\begin{aligned}
&\mathrm{eig}_{\max}(\mathbf{F}) \\
&\quad \leq |||\mathbf{F}|||_{\infty} && (40)\\
&\quad = \max_i \sum_j \left| \left[\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{H}\mathbf{G}(\lambda_i)^{\dagger}\mathbf{H}^H\mathbf{\Lambda}^{\frac{1}{2}}\right]_{i,j}^2 \right| && (41)\\
&\quad = \max_i \left[\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{H}\mathbf{G}(\lambda_i)^{\dagger}\mathbf{H}^H\mathbf{\Lambda}\mathbf{H}\mathbf{G}(\lambda_i)^{\dagger}\mathbf{H}^H\mathbf{\Lambda}^{\frac{1}{2}}\right]_{i,i} && (42)\\
&\quad = \max_i \Big\{ \left[\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{H}\mathbf{G}(\lambda_i)^{\dagger}\mathbf{H}^H\mathbf{\Lambda}^{\frac{1}{2}}\right]_{i,i} \\
&\qquad - \left[\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{H}\mathbf{G}(\lambda_i)^{\dagger}\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}\mathbf{G}(\lambda_i)^{\dagger}\mathbf{H}^H\mathbf{\Lambda}^{\frac{1}{2}}\right]_{i,i} \Big\} && (43)\\
&\quad \leq \max_i \left\{ \left[\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{H}\mathbf{G}(\lambda_i)^{\dagger}\mathbf{H}^H\mathbf{\Lambda}^{\frac{1}{2}}\right]_{i,i} \right\} && (44)\\
&\quad = \left[\mathbf{H}\mathbf{G}(\lambda_i)\mathbf{H}^H\mathbf{\Lambda}\right]_{i,i} && (45)\\
&\quad = \frac{\gamma_o}{1+\gamma_o} && (46)
\end{aligned}$$

where the inequality in (40) stems from the fact that any induced matrix norm upper bounds the maximal eigenvalue of the matrix. The equality in (41) is the definition of the row-sum-induced matrix norm. The inequality in (44) stems from neglecting the nonpositive terms in (43), and the equality in (45) is due to (35). We still need to prove that the inequality is strict, but this can be proven as follows. Assume that the inequality is not strict, i.e., there exists an i such that the second element in (43) is zero, i.e., $\mathbf{H}_{\mathrm{Tx}}\mathbf{G}(\lambda_i)^{\dagger}\mathbf{H}^H\mathbf{\Lambda}^{1/2}\mathbf{e}_i = \mathbf{0}$, and therefore $[\mathbf{\Lambda}^{1/2}\mathbf{H}_{\mathrm{RxCh}}\mathbf{H}_{\mathrm{Tx}}\mathbf{G}(\lambda_i)^{\dagger}\mathbf{H}^H\mathbf{\Lambda}^{1/2}]_{i,i} = 0$. However, since $\gamma_o > 0$, this a contradiction to (35), and therefore the inequality in (44) must be strict.

We now show that the arguments of the squared roots in (37) are nonnegative. Using a series expansion for the matrix inversion yields [50]

$$\begin{aligned}
\begin{bmatrix} \delta_1^2 \\ \vdots \\ \delta_K^2 \end{bmatrix} &= \left[\frac{\gamma_o}{1+\gamma_o}\mathbf{I} - \mathbf{F}\right]^{-1} \begin{bmatrix} \lambda_1\sigma_1^2 \\ \vdots \\ \lambda_K\sigma_K^2 \end{bmatrix} \\
&= \frac{1+\gamma_o}{\gamma_o} \sum_{j=1}^{\infty} \left[\frac{\gamma_o}{1+\gamma_o}\mathbf{F}\right]^j \begin{bmatrix} \lambda_1\sigma_1^2 \\ \vdots \\ \lambda_K\sigma_K^2 \end{bmatrix}.
\end{aligned} \tag{47}$$

The elements of $\gamma_o/(1+\gamma_o)\mathbf{F}$ are nonnegative. Therefore, the elements of the sum will also be nonnegative, and we can take the element wise squared roots and solve for δ_i for $i = 1, \ldots, K$.

Thus, we have shown that the solution in (36)–(38) exists. Plugging this solution into the SINR constraints satisfies all the constraints with equality. Therefore, the problem is feasible. Moreover, since $\sigma_i^2 > 0$ for $i = 1, \ldots, K$, we can always scale the solution $\mathbf{T}$ by $c > 1$ and satisfy the constraints with strict inequalities, i.e., the problem is strictly feasible.

In the next part of the proof, we will show that if (35) holds, then the solution in (36)–(38) is necessary and sufficient for optimality. The power optimization problem can be written as follows:

$$\mathcal{P}(\gamma_o): \begin{cases} \min_{\mathbf{T}} & \mathrm{Tr}\left\{\mathbf{T}^H\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}\mathbf{T}\right\} \\ \text{s.t.} & \left\| \begin{bmatrix} \mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i \end{bmatrix} \right\|^2 - \left(1 + \frac{1}{\gamma_o}\right)|[\mathbf{H}\mathbf{T}]_{i,i}|^2 \\ & \leq 0, \quad i = 1, \ldots, K. \end{cases} \tag{48}$$

The above program is not written in convex form (in order to write it in convex form, conic inequalities must be used). In general, the KKT conditions are not sufficient for optimality in nonconvex programs. However, in Appendix I, we show that in this special case, if the program is strictly feasible, then its KKT conditions are necessary and sufficient for optimality. The Lagrangian associated with program (48) is

$$\mathcal{L} = \mathrm{Tr}\left\{\mathbf{T}^H\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}\mathbf{T}\right\} + \sum_i \lambda_i \left[\left\|\begin{bmatrix}\mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i\end{bmatrix}\right\|^2 - \left(1+\frac{1}{\gamma_o}\right)|[\mathbf{HT}]_{i,i}|^2\right] \tag{49}$$

where $\lambda_i \geq 0$ are the Lagrange dual variables. As we have shown in the first part of the proof, if (35) holds, then the problem is strictly feasible. Therefore, its primal and dual variables are optimal if and only if the following conditions are satisfied.

1) *Feasibility*: The variable $\mathbf{T}$ is feasible

$$\left(1+\frac{1}{\gamma_o}\right)|[\mathbf{HT}]_{i,i}|^2 \geq \left\|\begin{bmatrix}\mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i\end{bmatrix}\right\|^2 \tag{50}$$

for $i = 1,\ldots,K$, and the dual variables are dual feasible, i.e., $\lambda_i \geq 0$ for $i = 1,\ldots,K$.

2) *Complementary slackness*: For each $i = 1,\ldots,K$, either $\lambda_i = 0$ or

$$\left(1+\frac{1}{\gamma_o}\right)|[\mathbf{HT}]_{i,i}|^2 = \left\|\begin{bmatrix}\mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i\end{bmatrix}\right\|^2. \tag{51}$$

3) *Zero derivative*: The derivative of $\mathcal{L}$ with respect to $\mathbf{T}$ is zero, resulting in

$$\mathbf{G}(\lambda_i)\mathbf{T} - \mathbf{H}^H\mathbf{\Lambda}^{\frac{1}{2}}\mathrm{diag}\left\{\left(1+\frac{1}{\gamma_o}\right)\left[\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{HT}\right]_{i,i}\right\} = \mathbf{0}. \tag{52}$$

At the optimal solution, all the constraints are active, i.e., (51) holds with equality for $i = 1,\ldots,K$. As proof, note that if one constraint does not hold with an equality, then we can always scale the row in $\mathbf{T}$ associated with it and arrive with a feasible solution that results in a lower objective value, which is a contradiction.

Another important property of the optimal solution, is that all the dual variables are strictly positive. As proof, assume the contrary, i.e., there exists an i such that $\lambda_i = 0$. Then, multiplying (52) by $\mathbf{T}^H$ on the left and examining the ith diagonal element, we have $\mathbf{e}_i^H\mathbf{T}^H\mathbf{G}(\lambda_i)\mathbf{T}\mathbf{e}_i = 0$, which holds if and only if $\mathbf{\Lambda}^{1/2}\mathbf{HT}\mathbf{e}_i = \mathbf{0}$ and $\mathbf{H}_{\mathrm{Tx}}\mathbf{T}\mathbf{e}_i = \mathbf{0}$, in which case the ith SINR is clearly zero. However, since $\gamma_o > 0$, this contradicts the SINR constraints.

In general, the $\mathbf{T}$ that satisfies (52) is not unique. Nonetheless, expressing $\mathbf{T}$ as $\mathbf{T} = \mathbf{T}_{\|} + \mathbf{T}_{\perp}$, where $\mathbf{T}_{\|} = \mathbf{PT}, \mathbf{T}_{\perp} = (\mathbf{I}-\mathbf{P})\mathbf{T}$, and $\mathbf{P} = \mathbf{H}_{\mathrm{Tx}}^{\dagger}\mathbf{H}_{\mathrm{Tx}}$, we can find $\mathbf{T}_{\|}$, which is unique within the range of $\mathbf{H}_{\mathrm{Tx}}^H$. Using $\mathbf{HT} = \mathbf{HT}_{\|}$, we arrive with the following necessary and sufficient conditions:

$$\mathbf{T}_{\|} = \mathbf{G}^{\dagger}(\lambda_i)\mathbf{H}^H\mathbf{\Lambda}^{\frac{1}{2}}\mathrm{diag}\{\delta_i\} \tag{53}$$

$$\left(1+\frac{1}{\gamma_o}\right)\left|\left[\mathbf{HT}_{\|}\right]_{i,i}\right|^2 = \left\|\begin{bmatrix}\mathbf{T}_{\|}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i\end{bmatrix}\right\|^2 \tag{54}$$

where $\delta_i = (1+1/\gamma_o)[\mathbf{\Lambda}^{1/2}\mathbf{HT}_{\|}]_{i,i}$ for $i = 1,\ldots,K$. As already explained, if (35) holds, then the solution in (36)–(38) satisfies (54). In addition, it has the structure of (53) and is therefore sufficient. Moreover, it is easy to show that this structure is also necessary (within the range of $\mathbf{H}_{\mathrm{Tx}}^H$). Plugging $\mathbf{T}_{\|}$ from (53) into (54) yields

$$\left(1+\frac{1}{\gamma_o}\right)\frac{1}{\lambda_i}[\mathbf{F}]_{i,i}\delta_i^2 = \sum_j \frac{1}{\lambda_i}[\mathbf{F}]_{i,j}\delta_j^2 + \sigma_i^2 \tag{55}$$

for $i = 1,\ldots K$, where $\mathbf{F}$ is the matrix defined by (38). Rewriting in matrix form, we have

$$\left[\frac{\gamma_o}{1+\gamma_o}\mathbf{I} - \mathbf{F}\right]\begin{bmatrix}\delta_1^2 \\ \vdots \\ \delta_K^2\end{bmatrix} = \begin{bmatrix}\lambda_1\sigma_1^2 \\ \vdots \\ \lambda_K\sigma_K^2\end{bmatrix}. \tag{56}$$

Since $[\mathbf{HT}]_{i,i} \geq 0$ for $i = 1,\ldots,K$, the unique solution to this set of equations is given by (37) and (38). Finally, the optimal objective value in (39) can be easily found using (5) and (52).

For completeness, it should be noted that when the problem is solvable, there always exist λ_i for $i = 1,\ldots,K$ such that (35) holds. This can be shown since if we left multiply both sides of (53) by $\mathbf{H}$ and examine the diagonal elements, then (35) is a direct consequence of (53) (which is a necessary condition for optimality). ■

Theorem 1 provides a simple strategy for designing the precoder. Given a feasible γ_o, all one has to do is find $\lambda_i > 0$ which satisfy (35). Once these are found, $\mathbf{T}$ can be derived through (36)–(38). As we will show in Section VI, in some special cases, these variables can be derived in closed form. Otherwise, we now propose two alternative methods for finding these variables. In Section IV-C-1), we present a simple fixed-point iteration, and in Section IV-C-2), we propose an SDP dual program.

1) Fixed-Point Iteration for Finding λ_i: The structure of (35) motivates a fixed-point iteration for finding λ_i. By rearranging (35), we arrive at the following simple iteration:

$$\lambda_i^{(n+1)} = \frac{\gamma_o}{1+\gamma_o}\frac{1}{\left[\mathbf{H}\mathbf{G}^{\dagger}\left(\lambda_i^{(n)}\right)\mathbf{H}^H\right]_{i,i}}, \quad i = 1,\ldots K. \tag{57}$$

Clearly, the optimal λ_i satisfy this fixed point. As we now show, if $\boldsymbol{\mathcal{P}}(\gamma_o)$ is feasible, then the above iteration will converge from any $\lambda_i^{(0)}$ to a set $\lambda_i^{(n)} > 0$ that satisfies (35). The convergence proof is based on the *standard function* approach introduced in [51], which can be summarized as follows. Consider the fixed-point iteration

$$\lambda_i^{(n+1)} = f_i\left(\mathbf{\Lambda}^{(n)}\right), \quad i = 1,\ldots K \tag{58}$$

where $\mathbf{\Lambda}^{(n)} = \mathrm{diag}\{\lambda^{(n)}\}$. If (58) has a fixed point, and the functions $f_i(\mathbf{\Lambda})$ obey the following properties:

- (*positivity*) if $\lambda_i \geq 0$ for all i, then $f_j(\mathbf{\Lambda}) > 0$ for all j;
- (*monotonicity*) if $\lambda_i \geq \lambda_i'$ for all i, then $f_j(\mathbf{\Lambda}) \geq f_j(\mathbf{\Lambda}')$ for all j;
- (*scalability*) if $\alpha > 1$, then $\alpha f_j(\mathbf{\Lambda}) > f_j(\alpha\mathbf{\Lambda})$ for all j

then starting in any initial $\mathbf{\Lambda}^{(0)}$, the iteration will converge to this unique fixed point. In Appendix II, we show that if the problem is feasible and $[\mathbf{H}(\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}})^{\dagger}\mathbf{H}^H]_{i,i} < \infty$ for $i = 1,\ldots,K$, then the functions in (57) satisfy these properties, and the iteration will converge.

2) Dual Program for Finding λ_i: Alternatively, the dual variables can be found through a dual program. The dual program is a concave program that optimizes the dual variables. Due to space limitations, the details of its derivation are omitted. The resulting program is

$$\mathcal{P}_{\mathcal{D}}(\gamma_o): \begin{cases} \max_{\lambda_i \geq 0} & \sum_i \lambda_i \sigma_i^2 \\ \text{s.t.} & \frac{\gamma_o}{1+\gamma_o}\mathbf{G}(\lambda_i) - \lambda_i \mathbf{H}^H \mathbf{e}_i \mathbf{e}_i^H \mathbf{H} \succeq 0, \\ & i = 1, \ldots, K. \end{cases} \tag{59}$$

This is a simple SDP/LMI program, which can be efficiently solved by any standard SDP/LMI optimization package. Moreover, it has only K optimization variables, in comparison to K^2 optimization variables in the original program, and therefore has a lower computational complexity. A similar result was obtained in [39] in the context of beamforming.

D. Interpretation via Uplink–Downlink Duality

In this section, we provide an alternative solution for the power optimization problem based on the well known uplink–downlink duality [40], [41]. As explained in the previous sections, the problem can be solved efficiently without the use of duality. However, previous attempts for solving the downlink beamforming problem, which is a special case of precoding (where $\mathbf{H}_{\text{Tx}} = \mathbf{I}$), are based on this approach. Therefore, for completeness, we now review this method and generalize it to the case of precoding, i.e., arbitrary $\mathbf{H}_{\text{Tx}}$. Moreover, the duality is interesting from an engineering point of view, as it provides an interesting physical interpretation of the solution.

Recently, an interesting duality was found between downlink beamforming and another problem called uplink beamforming. It is usually referred to as downlink (broadcast)–uplink (multiple access) duality, since one problem typically arises in the broadcast channel of a downlink system, and the other arises in the multiple access channel of an uplink system. Fortunately, the uplink beamforming problem is easier to solve. Using the duality, the downlink solution can be derived through the uplink solution. For simplicity, in the sequel, we restrict ourselves to full-rank channels. Mathematically, the duality can be stated as follows.

Theorem 2: Consider the *uplink* program in (60), shown at the bottom of the page. Program $\bar{\mathcal{P}}(\gamma_o)$ is the dual of the program $\mathcal{P}(\gamma_o)$ of (6) in the sense that if the optimal arguments and objective value of $\bar{\mathcal{P}}$ are $\mathbf{W}, \lambda_i$, and P_o, then the optimal objective value of $\mathcal{P}$ is also P_o, and its optimal argument is $\mathbf{T} = \mathbf{W}^H \text{diag}\{\theta_i\}$, where θ_i are appropriate scaling coefficients.

Proof: It is easy to see that each constraint in $\bar{\mathcal{P}}$ deals with one row of $\mathbf{W}$ and that the objective is not a function of $\mathbf{W}$ at all. Therefore, it is clear that each row of $\mathbf{W}$ will be chosen to maximize the SINR associated with it. Thus, for fixed λ_i, the optimal receiver $\mathbf{W}$ is the well-known scaled MMSE matrix [1]

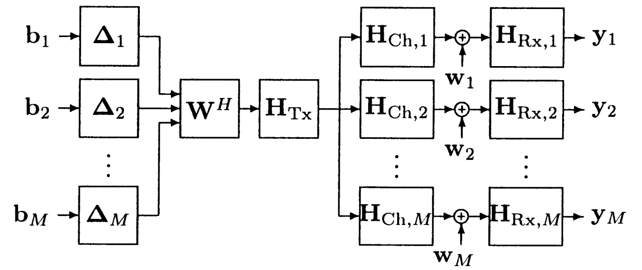

Fig. 2. Block diagram of a downlink (broadcast) system. The matrices $\mathbf{\Delta}_m$ for $m = 1, \ldots, M$ are diagonal matrices with the θ_is associated with $\mathbf{b}_m$.

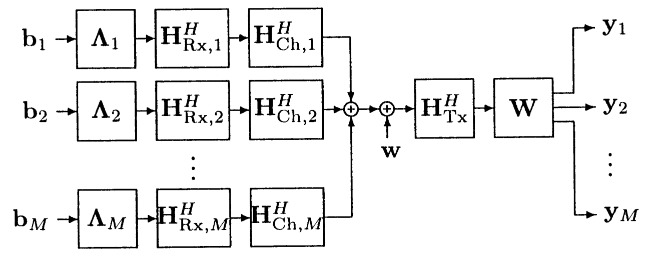

Fig. 3. Block diagram of a uplink (multiple access) system. The matrices $\mathbf{\Lambda}_m$ for $m = 1, \ldots, M$ are diagonal matrices with the λ_i's associated with $\mathbf{b}_m$. The vector $\mathbf{w}$ is the virtual uplink noise vector.

$$\mathbf{W} = \mathbf{H}\mathbf{G}^{-1}(\lambda_i) \tag{61}$$

which is unique up to a diagonal matrix multiplication on the left. In addition, similarly to the downlink problem, all the constraints of the uplink problem are active (otherwise, one can always decrease the λ_i associated with the passive constraint and decrease the objective). Thus, at the optimum

$$\frac{\lambda_i |[\mathbf{W}\mathbf{H}^H]_{i,i}|^2}{\sum_{j \neq i} \lambda_j |[\mathbf{W}\mathbf{H}^H]_{i,j}|^2 + \left[\mathbf{W}\mathbf{H}_{\text{Tx}}^H \mathbf{H}_{\text{Tx}} \mathbf{W}^H\right]_{i,i}} = \gamma_o \tag{62}$$

for $i = 1, \ldots, K$. Plugging in the optimal $\mathbf{W}$ and simplifying the terms results in

$$\lambda_i [\mathbf{H}\mathbf{G}^{-1}(\lambda_i)\mathbf{H}^H]_{i,i} = \frac{\gamma_o}{1+\gamma_o}, \quad i = 1, \ldots, K. \tag{63}$$

Thus, the optimal λ_i's of $\bar{\mathcal{P}}$ satisfy (35), and by appropriately choosing θ_i, the precoder $\mathbf{T} = \mathbf{W}^H \text{diag}\{\theta_i\}$ satisfies also (36). For example, if the optimal $\mathbf{W}$ is scaled as in (61), then $\theta_i = \lambda_i^{1/2} \delta_i$. Therefore, according to Theorem 1, this precoder is optimal for $\mathcal{P}$. ∎

This uplink–downlink duality was originally developed for the special case of $\mathbf{H}_{\text{Tx}} = \mathbf{I}$. In Theorem 2, we generalize this result to arbitrary $\mathbf{H}_{\text{Tx}}$. The importance of this theorem is in its interesting interpretation of the optimal solution. It provides a physical interpretation to the positive dual variables $\lambda_i > 0$ as the virtual normalized power allocation. In order to visualize this duality, we provide block diagrams of the two dual systems in Figs. 2 and 3.

$$\bar{\mathcal{P}}(\gamma_o) = \begin{cases} \min_{\lambda_i > 0, \mathbf{W}} & \sum_i \lambda_i \sigma_i^2 \\ \text{s.t.} & \frac{\lambda_i |[\mathbf{W}\mathbf{H}^H]_{i,i}|^2}{\sum_{j \neq i} \lambda_j |[\mathbf{W}\mathbf{H}^H]_{i,j}|^2 + \left[\mathbf{W}\mathbf{H}_{\text{Tx}}^H \mathbf{H}_{\text{Tx}} \mathbf{W}^H\right]_{i,i}} \geq \gamma_o, \qquad i = 1, \ldots, K. \end{cases} \tag{60}$$

Moreover, previous attempts for solving the $\boldsymbol{\mathcal{P}}$ problem are based on this duality [38], [42]. As we have shown in the previous section, the problem can be solved without the duality using the optimality conditions. However, for completeness, we now present the duality-based approach as well. This approach confronts the $\boldsymbol{\mathcal{P}}$ problem by addressing the $\bar{\boldsymbol{\mathcal{P}}}$ problem first and then adjusting the solution based on Theorem 2. Fortunately, there is an intuitive iterative solution to program $\bar{\boldsymbol{\mathcal{P}}}$. The problem can be solved by iteratively solving for each of the parameters, while keeping the others fixed.

$\bar{\boldsymbol{\mathcal{P}}}(\gamma_o)$

1 **repeat**

2 $\quad \mathbf{W} \leftarrow \mathbf{H}\mathbf{G}^{-1}(\lambda_i)$

3 $\quad \lambda_i \leftarrow \gamma_o \frac{\sum_{j\neq i} \lambda_j |[\mathbf{W}\mathbf{H}^H]_{i,j}|^2 + \|\mathbf{H}_{\mathrm{Tx}}\mathbf{W}^H\mathbf{e}_i\|^2}{|[\mathbf{W}\mathbf{H}^H]_{i,i}|^2}$ $\quad i = 1, \ldots, K.$

4 **until** convergence

Line 2 optimizes the receive matrix $\mathbf{W}$ to maximize the SINRs for fixed λ_i. Line 3 optimizes the power allocation weights λ_i for fixed $\mathbf{W}$ [51]. In [38], it was shown that the above algorithm always converges to the optimal solution. It is similar to our simple fixed-point iteration in (57), except for the fact that in the above algorithm $\mathbf{W}$ and λ_i are independently optimized at each iteration, whereas in (57) both are optimized together. Thus, our simple fixed-point iteration is more appealing.

V. SINR Optimization

We now consider the problem of maximizing the worst SINR subject to a power constraint, i.e., the $\boldsymbol{\mathcal{S}}$ program of (7). As before, we begin by examining its feasibility. Fortunately, it is easy to verify that the $\boldsymbol{\mathcal{S}}$ program is always feasible, as we can always scale $\mathbf{T}$ so that it satisfies the power constraint. In Section V-A, we discuss the connection between the power optimization and the SINR optimization and explain how this connection can be used to solve the SINR optimization. Then, we follow the steps we took before in the context of the power optimization and repeat them in the context of the SINR optimization. In Section V-B, we formulate the SINR problem as a standard GEVP conic program; in Section V-C, we provide a fixed point iteration; and in Section V-D, we discuss its uplink–downlink duality.

A. Connection With Power Optimization

The most interesting property of the SINR optimization program is its relation to the power optimization program. In order to mathematically define this relation, we introduce the following theorem.

Theorem 3: The power optimization problem of (6) and the SINR optimization problem of (7) are inverse problems:

$$\gamma_o = \boldsymbol{\mathcal{S}}(\boldsymbol{\mathcal{P}}(\gamma_o)) \tag{64}$$

$$P_o = \boldsymbol{\mathcal{P}}(\boldsymbol{\mathcal{S}}(P_o)). \tag{65}$$

In addition, the optimal objective value of each program is continuous and strictly monotonic increasing in its input argument

$$\gamma_o > \tilde{\gamma}_o \Rightarrow \boldsymbol{\mathcal{P}}(\gamma_o) > \boldsymbol{\mathcal{P}}(\tilde{\gamma}_o) \tag{66}$$

$$P_o > \tilde{P}_o \Rightarrow \boldsymbol{\mathcal{S}}(P_o) > \boldsymbol{\mathcal{S}}(\tilde{P}_o). \tag{67}$$

Proof: We begin by proving (64) by contradiction. Assume the contrary, i.e., P and $\mathbf{T}$ are the optimal value and argument of $\boldsymbol{\mathcal{P}}(\gamma)$, and $\tilde{\gamma} \neq \gamma$ and $\tilde{\mathbf{T}}$ are the optimal value and argument of $\boldsymbol{\mathcal{S}}(P)$. If $\tilde{\gamma} < \gamma$, then this is a contradiction for the optimality of $\tilde{\mathbf{T}}$ for $\boldsymbol{\mathcal{S}}(P)$, since $\mathbf{T}$ is feasible for it and provides a larger objective value γ. Otherwise, if $\tilde{\gamma} > \gamma$, then this is a contradiction for the optimality of $\mathbf{T}$ for $\boldsymbol{\mathcal{P}}(\gamma)$, since $\tilde{\gamma} > \gamma$, and we can always find $c < 1$ such that $c\tilde{\mathbf{T}}$ will still be feasible but will result in a smaller objective.

Next, we prove (66) by contradiction. Assume the contrary, i.e., P and $\mathbf{T}$ are optimal for γ, and $\tilde{P} \geq P$ and $\tilde{\mathbf{T}}$ are optimal for $\tilde{\gamma} < \gamma$. We can always multiply $\mathbf{T}$ by $c < 1$ so that it will still achieve the SINRs constraints of $\tilde{\gamma}$, with an effective power constraint $c^2 P < P \leq \tilde{P}$. This contradicts the assumption that $\tilde{\mathbf{T}}$ was optimal for $\tilde{\gamma}$. The continuity can be verified using similar arguments to those in Lemma 2 of [52]. The proofs of (65) and (67) are similar and are therefore omitted. ■

Using the properties in Theorem 3, we can solve $\boldsymbol{\mathcal{S}}(P_o)$ for a given P_o by iteratively solving $\boldsymbol{\mathcal{P}}(\gamma_o)$ for different γ_os. Due to the inversion property, if $P_o = \boldsymbol{\mathcal{P}}(\gamma_o)$, then its solution will be optimal also for $\boldsymbol{\mathcal{S}}(P_o)$. The strict monotonicity and continuity guarantees that a simple one-dimensional bisection search will efficiently find the required γ_o. This procedure is summarized in the following algorithm (see also [43]).

$\boldsymbol{\mathcal{S}}(P_o)$

$\gamma_{\max} \leftarrow \mathrm{MaxSINR}$

$\gamma_{\min} \leftarrow \mathrm{MinSINR}$

repeat

$\quad \gamma_o \leftarrow (\gamma_{\min} + \gamma_{\max})/2$

$\quad \hat{P}_o \leftarrow \boldsymbol{\mathcal{P}}(\gamma_o)$

$\quad$ **if** $\hat{P}_o \leq P_o$

$\quad\quad$ **then** $\gamma_{\min} \leftarrow \gamma_o$

$\quad\quad$ **else** $\gamma_{\max} \leftarrow \gamma_o$

$\quad$ **until** $\hat{P}_o = P_o$

return γ_o

where MinSINR and MaxSINR define a range of relevant SINRs for a specific application, and where we have used the convention that $\infty = \boldsymbol{\mathcal{P}}(\gamma_o)$ if it is infeasible.

Theoretically, this means that the SINR optimization problem can be solved through the previous results concerning the power optimization. Nonetheless, due to its importance and in order to obtain more efficient numerical solution, we now provide direct solutions for the SINR optimization through conic optimization, via the optimality conditions, and through the uplink–downlink duality.

B. Conic Optimization Solution

The SINR optimization can be cast as a standard GEVP program. Using a real-valued slack variable γ_o, the problem can be rewritten as

$$\boldsymbol{\mathcal{S}}(P_o): \begin{cases} \max_{\mathbf{T},\gamma_o} & \gamma_o \\ \text{s.t.} & \frac{|[\mathbf{H}\mathbf{T}]_{i,i}|^2}{\sum_{j\neq i}|[\mathbf{H}\mathbf{T}]_{i,j}|^2 + \sigma_i^2} \geq \gamma_o, \\ & \qquad i = 1, \ldots, K \\ & \mathrm{Tr}\left\{\mathbf{T}^H\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}\mathbf{T}\right\} \leq P_o. \end{cases} \tag{68}$$

At first glance, (68) seems similar to (26). However, it turns out to be considerably more complicated. This is because the SINR matrix inequalities in (33) are linear in $\beta = \sqrt{1 + 1/\gamma_o}$

or in $\mathbf{T}$, but not in both simultaneously. Thus, when β is an optimization variable and not a parameter, these constraints are no longer LMIs. In fact, the sets which they define are not convex.[3] Nonetheless, we can still express them using generalized matrix inequalities as in (32) and (33). If we rewrite the $\mathbf{A}_i(\mathbf{T})$'s in (33) and separate out the terms which are linear, we have

$$\mathbf{A}_i(\mathbf{T}) = \beta \mathbf{A}_i^1(\mathbf{T}) - \mathbf{A}_i^2(\mathbf{T}) \tag{69}$$

where $\mathbf{A}_i^1(\mathbf{T})$ and $\mathbf{A}_i^2(\mathbf{T})$ are matrices that depend affinely on $\mathbf{T}$, as follows:

$$\mathbf{A}_i^1(\mathbf{T}) = \begin{bmatrix} [\mathbf{HT}]_{i,i} & \mathbf{0} \\ \mathbf{0} & [\mathbf{HT}]_{i,i}\mathbf{I} \end{bmatrix};$$
$$\mathbf{A}_i^2(\mathbf{T}) = \begin{bmatrix} 0 & -\begin{bmatrix} \mathbf{e}_i^H\mathbf{HT} & \sigma_i \end{bmatrix} \\ -\begin{bmatrix} \mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i \end{bmatrix} & \mathbf{0} \end{bmatrix}. \tag{70}$$

Using (69), we can express $\boldsymbol{\mathcal{S}}$ in the standard GEVP form

$$\boldsymbol{\mathcal{S}}(P_o): \begin{cases} \min_{\mathbf{T},\beta} & \beta \\ \text{s.t.} & \beta\mathbf{A}_i^1(\mathbf{T}) \succeq \mathbf{A}_i^2(\mathbf{T}), \quad i=1,\ldots,K \\ & \mathbf{A}_i^1(\mathbf{T}) \succeq 0, \quad i=1,\ldots,K \\ & \mathbf{C}(\mathbf{T}) \succeq 0 \end{cases} \tag{71}$$

which can be solved using appropriate software [49].

C. Fixed-Point Iteration for Finding Λ_i

The SINR optimization problem can also be solved using the conditions in Theorem 1. As explained in Theorem 3, $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{P}}$ are inverse problems. Thus, the optimal solution of the SINR optimization is also optimal for an inverse power optimization problem, and therefore must satisfy its optimality conditions as well. Thus, to optimize the SINRs, we need to find $\lambda_i > 0$ that satisfy (35) and (39). Unfortunately, in this case, γ_o is an optimization variable and not a parameter and has to be found as well. This can be overcome by adjusting the fixed-point iteration in (57), as follows:

$$\tilde{\lambda}_i = \frac{1}{\left[\mathbf{HG}^\dagger\left(\lambda_i^{(n)}\right)\mathbf{H}^H\right]_{i,i}}, \quad i=1,\ldots,K \tag{72}$$

and then normalizing the result so that it will satisfy (39):

$$\lambda_i^{(n+1)} = \frac{P_o\tilde{\lambda}_i}{\sum_j \sigma_j^2\tilde{\lambda}_j}, \quad i=1,\ldots K. \tag{73}$$

If this iteration converges to a fixed point $\lambda_i^{(n)} > 0$, then it will satisfy (35) and (39). Numerous numerical simulations with arbitrary initial points and parameters show a rapid convergence rate.

[3]The exact definition of such sets is quasi-convex [21].

D. Interpretation via Uplink–Downlink Duality

Following the success of the uplink–downlink duality in the power optimization, the duality was recently used to confront the SINR optimization [44]. The uplink–downlink duality in the case of the SINR optimization can be stated as follows.

Theorem 4: Consider the *uplink* program in (74), shown at the bottom of the page. Program $\bar{\boldsymbol{\mathcal{S}}}(P_o)$ is the dual of the program $\boldsymbol{\mathcal{S}}(P_o)$ of (7) in the sense that if the optimal arguments and objective value of $\bar{\boldsymbol{\mathcal{S}}}$ are $\mathbf{W}, \lambda_i$ and γ_o, then the optimal objective value of $\boldsymbol{\mathcal{S}}$ is also γ_o, and its optimal argument is $\mathbf{T} = \mathbf{W}^H\text{diag}\{\theta_i\}$, where θ_i are appropriate scaling coefficients.

Proof: The proof is similar to the other proofs in this paper and is therefore omitted. A detailed proof in the case of $\mathbf{H}_{\text{Tx}} = \mathbf{I}$ can be found in [44]. ■

The downlink beamforming SINR optimization problem was solved using duality in [44]. The algorithm iteratively optimizes each of the optimization variables while keeping the others fixed, as follows.

$\bar{\boldsymbol{\mathcal{S}}}(P_o)$
1 **repeat**
2 $\quad \mathbf{W} \leftarrow \mathbf{HG}^{-1}(\lambda_i)$
3 $\quad \begin{bmatrix} \boldsymbol{\lambda} \\ 1 \end{bmatrix} \leftarrow \text{eig}_{\max}\left(\begin{bmatrix} \mathbf{Q} & \mathbf{q} \\ \frac{1}{P_o}\mathbf{v}^T\mathbf{Q} & \frac{1}{P_o}\mathbf{v}^T\mathbf{q} \end{bmatrix}\right)$
4 **until** convergence

where $\mathbf{Q}$ is a $K \times K$ matrix with elements $[\mathbf{Q}]_{i,j} = |[\mathbf{WH}^H]_{i,j}|^2/|[\mathbf{WH}^H]_{i,i}|^2, \mathbf{q}$ is a length K vector with elements $\mathbf{q}_i = \|\mathbf{H}_{\text{Tx}}\mathbf{W}^H\mathbf{e}_i\|^2/|[\mathbf{WH}^H]_{i,i}|^2$, and $\mathbf{v}$ is a length K vector with elements σ_i^2. Line 2 optimizes the matrix $\mathbf{W}$ for fixed λ_i. Line 3 optimizes the weights λ_i for fixed $\mathbf{W}$ based on [53]. Clearly, this solution is much less appealing than the fixed point iteration in (72) and (73).

VI. Special Cases

In this section, we examine a few interesting cases in which the problems $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{S}}$ have simple closed-form solutions.

A. Diagonal Case

The first case is when the matrices $\mathbf{H}$ and $\mathbf{H}_{\text{Tx}}$ are diagonal. In this case, it is trivial to satisfy the optimality conditions in Theorem 1. The resulting precoders are diagonal and can be considered as simple power allocation strategies.

B. Symmetric Case

The second case is when the matrices $\mathbf{H}$ and $\mathbf{H}_{\text{Tx}}$ have equal diagonal elements and equal off-diagonal elements, and the variances are equal $\sigma_i^2 = \sigma^2$. Due to the symmetry, it is clear that choosing $\lambda_i = P_o/(K\sigma^2)$ will satisfy the conditions in

$$\bar{\boldsymbol{\mathcal{S}}}(P_o) = \begin{cases} \max_{\mathbf{W},\lambda_i\geq 0,\gamma_o} & \gamma_o \\ \text{s.t.} & \frac{\lambda_i|[\mathbf{WH}^H]_{i,i}|^2}{\sum_{j\neq i}\lambda_j|[\mathbf{WH}^H]_{i,j}|^2 + \left[\mathbf{WH}_{\text{Tx}}^H\mathbf{H}_{\text{Tx}}\mathbf{W}^H\right]_{i,i}} \geq \gamma_o, \quad i=1,\ldots,K \\ & \sum_i \sigma_i^2\lambda_i \leq P_o. \end{cases} \tag{74}$$

Theorem 1. Therefore, the solution for the SINR optimization problem is

$$\gamma_o = \frac{1}{\frac{1}{\left[\mathbf{H}\left(\mathbf{H}^H\mathbf{H}+\frac{K\sigma^2}{P_o}\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}\right)^{-1}\mathbf{H}^H\right]_{i,i}} - 1} \tag{75}$$

$$\mathbf{T} = c\left[\mathbf{H}^H\mathbf{H} + \frac{K\sigma^2}{P_o}\mathbf{H}_{\mathrm{Tx}}^H\mathbf{H}_{\mathrm{Tx}}\right]^{\dagger}\mathbf{H}^H \tag{76}$$

where c is a constant that scales the matrix to satisfy the power constraint. This particular precoder has been previously derived in [5]–[9] through scaled MMSE considerations. However, it is easy to verify that in general, i.e., in nonsymmetric channels, it does not necessarily satisfy the conditions in Theorem 1 and is therefore suboptimal in this sense. For example, if the channels are symmetric but the noise variances are not equal, then, in order to ensure equal SINRs among all the streams, the precoder in (76) must be diagonally scaled using (37).

VII. Applications in Multiuser Systems

In this section, we present possible applications of the proposed precoders to multiuser systems. Consider a multiuser precoded downlink system. At each symbol's period, the base station transmits using an $N \times K$ nonorthogonal signatures matrix $\mathbf{H}_{\mathrm{Tx}} = \mathbf{S}$. The maximal average transmitted power is $P_o = K$, and the cross correlations between the signatures are denoted by $\rho_{i,j} = [\mathbf{S}^H\mathbf{S}]_{i,j}$ with $\rho_{i,i} = 1$ for all i. For simplicity, we assume ideal channels, i.e., $\mathbf{H}_{\mathrm{Ch},i} = \mathbf{I}$, and equal noise variances, i.e., $\sigma_i^2 = \sigma^2$. Denoting by $\mathbf{y}$ the output vector of the multiple user receiver, we have that

$$\mathbf{y} = \mathbf{H}_{\mathrm{Rx}}\mathbf{S}\mathbf{T}\mathbf{b} + \mathbf{H}_{\mathrm{Rx}}\mathbf{w} \tag{77}$$

where $\mathbf{H}_{\mathrm{Rx}}$ is one of the standard filters, as follows:

- MF receiver: $\mathbf{H}_{\mathrm{Rx}} = \mathbf{S}^H$;
- ZF receiver: $\mathbf{H}_{\mathrm{Rx}} = (\mathbf{S}^H\mathbf{S})^{-1}\mathbf{S}^H$;
- MMSE receiver: $\mathbf{H}_{\mathrm{Rx}} = \mathbf{S}^H(\mathbf{S}\mathbf{S}^H + \sigma^2\mathbf{I})^{-1}$.

We now discuss the performance of these systems with and without precoding.

A. Equal Power and Equal Cross Correlations

The first interesting result of our precoder is its performance in an equal power and equal cross correlations multiuser system, i.e., $\rho_{i,j} = \rho$ for all $j \neq i$ and $\sigma_i^2 = \sigma^2$. As explained in Section VI-B, our precoder and its SINRs have closed forms in this case.

Proposition 2: Consider the multiuser system in (77). If $\mathbf{S}^H\mathbf{S}$ is invertible and $\rho_{i,j} = \rho$ for all $i \neq j$, then the output SINRs using the $\mathcal{S}$ precoder along with an MF receiver are identical to those resulting by using an MMSE receiver without any precoder and are equal to

$$\gamma_o = \frac{1}{\left[\left(\mathbf{I} + \frac{1}{\sigma^2}\mathbf{S}^H\mathbf{S}\right)^{-1}\right]_{i,i}} - 1. \tag{78}$$

Proof: The SINRs in (78) are obtained by applying the matrix inversion lemma on (75). In [1], it is shown that the output SINRs using MMSE receivers are also equivalent to (78). ∎

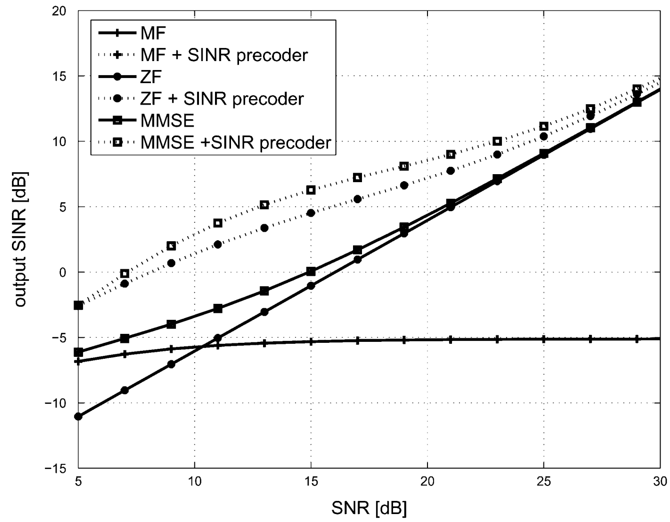

Fig. 4. SINR of a symmetric system with equal cross correlations. Due to the symmetry, all users have equal output SINRs.

Proposition 2 is interesting as it allows for each user to attain the MMSE performance without the use of an MMSE receiver which requires the knowledge of all the other signatures and a matrix inversion. Moreover, when the $\mathcal{S}$ precoder is used with ZF or MMSE receivers, the performance improves even more. In Fig. 4, we plot the output SINRs given by (75) for the three linear receivers. For comparison, we also plot the output SINRs that result from similar systems without a precoder [1]. As expected, using the precoder always improves the output SINR.

B. Nonsymmetric Channel

As a second example, we consider an equal power system with unequal cross correlations between the users signatures. In such systems, there is no closed-form expression for the performance. Therefore, we resort to Monte Carlo simulations. Following [3], we consider cross correlations $\rho_{12} = 0.8, \rho_{13} = 0.9$, and $\rho_{23} = 0.7$, where each user uses an MF receiver. For comparison, we provide results of the decorrelator precoder [3] and our SINR precoder of (7). Due to the asymmetry, each of the three users performs differently when using the previous precoders. On the other hand, our precoder has the attractive property of equal BERs for all the users. Naturally, the performance of the best user degrades compared with previous methods. In Fig. 5, we provide the BERs using each of the precoders. It is easy to see that the our precoder outperforms that of [3].

C. Rank-Deficient Channels

One of the main advantages of our precoder is its performance in rank-deficient systems. We now illustrate this property in a multiuser system with $K = 4$ users and length $N = 3$ sequences. The transmitter uses the optimal sequences of [34], and the receiver uses conventional matched filters. However, we use a distorting channel for the first user, i.e., $\mathbf{H}_{\mathrm{Ch},1}$ is a Toeplitz matrix with the first row $[1.0, 0.8, 0.0]$. Due to this channel, the sequences are no longer optimal, and a precoder should be used. The common decorrelating precoder of [3] cannot be derived in this case as $N < K$. Therefore, we compare our results to the precoder of [5]. The worst output SINRs with and without the

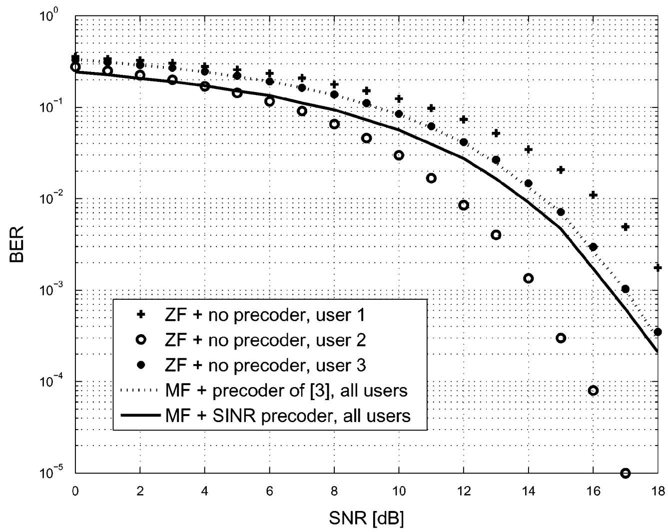

Fig. 5. BERs of a nonsymmetric three-user system with cross correlations $\rho_{12} = 0.8, \rho_{13} = 0.9, \rho_{23} = 0.7$.

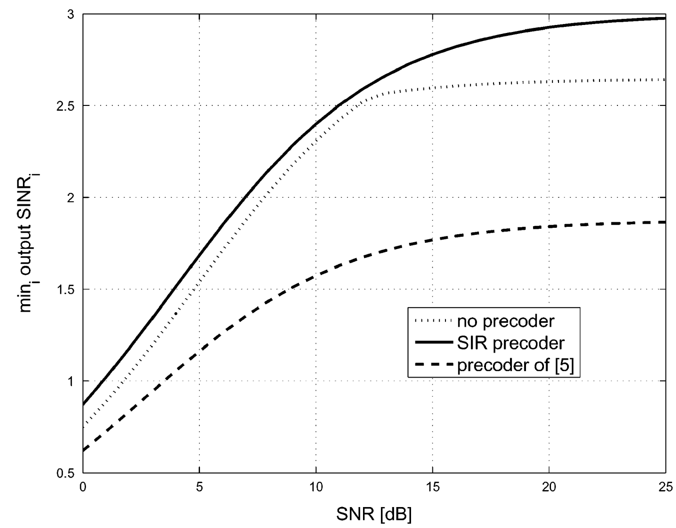

Fig. 6. Worst output SINR in a system with $K = 4$ and $N = 3$.

precoders are presented in Fig. 6. Using our $\mathcal{S}$ precoder significantly increases the SINR compared to a system with no precoder. Using the precoder, the SINRs asymptotically converge to the bound in (17), i.e., $\gamma_i = 1/(4/3 - 1) = 3$ for $i = 1, \ldots, 4$. Interestingly, the performance using the precoder of [5] is even worse than not using a precoder at all. For fairness, we must note that the SINR of the best user using this precoder are much higher. However, from a systems prespective, the interesting metric is the performance of the worst user, and in this sense our precoder is more appealing.

VIII. Conclusion

In this paper, we addressed the problem of designing linear precoders for fixed MIMO receivers. We considered two complementary design criteria, and proposed several alternative algorithmic solutions for these optimization problems.

It is first observed that in precoder design, maximizing the worst SINR is advantageous to minimizing MSE. Most of the previous work regarding precoders is based on optimizing variants of the MMSE criterion. These ad hoc criteria are usually computationally attractive and perform quite well. However, the ever increasing demand for better performance, as well as the considerable progress in optimization theory, suggests that upcoming research should focus on design criteria, which are more related to practical performance measures, such as maximizing the worst SINR.

Our second important observation is that by using conic optimization theory and algorithms, the precoder design problems can be solved in a straight forward manner without using uplink–downlink duality. This duality is remarkable and has enabled solutions to problems which were unsolvable before. Nonetheless, we believe that understanding the precoder design using first principles, and not as a byproduct of the uplink problem, is also important. For example, in future work, the simple optimality conditions may help in analyzing the performance of these systems, or in improving the design criteria, without the need to resort to the virtual uplink problem.

There are many interesting extensions to this paper that are worth pursuing. The first concerns the extension of our results to the case of partial channel state information (CSI). In many practical systems, the transmitter does not have access to perfect CSI, and needs to resort to noisy channel estimates, and/or delayed feedback. In this case, robust optimization algorithms should be applied. Another possible direction is to consider fixed nonlinear receivers, such as the successive canceling receiver. It is well known that such receivers outperform the linear receivers explored in our paper. Therefore, by designing the precoder to optimally work with such receivers, the performance can significantly improve.

Appendix I
Optimality Conditions for Programs With SOC Constraints

In this appendix, we derive optimality conditions for optimization programs with SOC constraints. The conditions are summarized in the following proposition.

Proposition 3: Consider a nonconvex program of the structure

$$\begin{cases} \min_{\mathbf{X}} & f(\mathbf{X}) \\ \text{s.t.} & \|\mathbf{a}_i(\mathbf{X})\|^2 - |a_i(\mathbf{X})|^2 \leq 0, \quad i = 1, \ldots, N \end{cases} \tag{79}$$

where $f(\mathbf{X})$ is convex in $\mathbf{X}$, and $a_i(\mathbf{X}) \geq 0$ and $\mathbf{a}_i(\mathbf{X})$ for $= 1, \ldots, K$ are affine functions of $\mathbf{X}$. Let us associate the dual variables λ_i for $i = 1, \ldots, K$ with this program. If the program is strictly feasible, then the following KKT conditions are necessary and sufficient conditions for optimality of $\mathbf{X}$ and λ_i.

1) *Feasibility*: The variable $\mathbf{X}$ is feasible

$$|a_i(\mathbf{X})|^2 \geq \|\mathbf{a}_i(\mathbf{X})\|^2, \quad i = 1, \ldots, K \tag{80}$$

and the dual variables are dual feasible $\lambda_i \geq 0$ for $i = 1, \ldots, K$.

2) *Complementary slackness*: For each $i = 1, \ldots, K$, one of the following conditions holds:

$$\lambda_i = 0 \quad \text{or} \quad |a_i(\mathbf{X})|^2 = \|\mathbf{a}_i(\mathbf{X})\|^2. \tag{81}$$

3) *Zero derivative*: The derivative of the Lagrangian of (79) with respect to $\mathbf{X}$ is zero

$$\frac{\partial}{\partial \mathbf{X}}\left\{f(\mathbf{X})+\sum_i \lambda_i[\|\mathbf{a}_i(\mathbf{X})\|^2-|a_i(\mathbf{X})|^2]\right\}=\mathbf{0}. \quad (82)$$

Proof: For simplicity, we will only deal with real-valued variables and functions. The extension to complex values is straightforward. The KKT conditions are necessary for optimality of any optimization problem [21]. If the program in (79) was in convex form, then the conditions were also sufficient for optimality. Unfortunately, the program is not expressed in convex form, and therefore we must prove the sufficiency. Let us begin by rewriting (79) in convex form

$$\begin{cases} \min_{\mathbf{X}} & f(\mathbf{X}) \\ \text{s.t.} & \begin{bmatrix} a_i(\mathbf{X}) \\ \mathbf{a}_i(\mathbf{X}) \end{bmatrix} \succeq_K 0, \quad i=1,\ldots,N. \end{cases} \quad (83)$$

If (83) is strictly feasible, then the following conic KKT conditions are necessary and sufficient for optimality [21].

1) *Feasibility*: The primal variable $\mathbf{X}$ is feasible, and the associated dual cones are dual feasible:

$$\begin{bmatrix} a_i(\mathbf{X}) \\ \mathbf{a}_i(\mathbf{X}) \end{bmatrix} \succeq_K 0, \quad \begin{bmatrix} w_i \\ \mathbf{w}_i \end{bmatrix} \succeq_K 0, \quad i=1,\ldots,K. \quad (84)$$

2) *Complementary slackness*:

$$\begin{bmatrix} w_i & \mathbf{w}_i^T \end{bmatrix}\begin{bmatrix} a_i(\mathbf{X}) \\ \mathbf{a}_i(\mathbf{X}) \end{bmatrix}=0, \quad i=1,\ldots,K. \quad (85)$$

3) *Zero derivative*: The derivative of the Lagrangian[4] of (83) with respect to $\mathbf{X}$ is zero

$$\frac{\partial}{\partial \mathbf{X}}\left\{f(\mathbf{X})-\sum_i\left(\begin{bmatrix} w_i & \mathbf{w}_i^T \end{bmatrix}\begin{bmatrix} a_i(\mathbf{X}) \\ \mathbf{a}_i(\mathbf{X}) \end{bmatrix}\right)\right\}=\mathbf{0}. \quad (86)$$

We now show that the conditions in (80)–(82) are sufficient for satisfying the conditions in (84)–(86). Let us choose

$$\begin{bmatrix} w_i \\ \mathbf{w}_i \end{bmatrix}=2\lambda_i\begin{bmatrix} a_i(\mathbf{X}) \\ -\mathbf{a}_i(\mathbf{X}) \end{bmatrix}, \quad i=1,\ldots,K. \quad (87)$$

Plugging the dual variables from (87) into the conic KKT conditions reveals that conditions (84) and (85) hold due to (80) and (81). Similarly, using (87), the conditions in (82) and (86) are identical, as follows:

$$\begin{aligned}&\frac{\partial}{\partial \mathbf{X}}\left\{f(\mathbf{X})-\sum_i\left(\begin{bmatrix} w_i & \mathbf{w}_i^T \end{bmatrix}\begin{bmatrix} a_i(\mathbf{X}) \\ \mathbf{a}_i(\mathbf{X}) \end{bmatrix}\right)\right\}\\ &=\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}}-\sum_i\left(2\lambda_i a_i(\mathbf{X})\frac{\partial a_i(\mathbf{X})}{\partial \mathbf{X}}-2\lambda_i\mathbf{a}_i^T(\mathbf{X})\frac{\partial \mathbf{a}_i(\mathbf{X})}{\partial \mathbf{X}}\right)\\ &=\frac{\partial}{\partial \mathbf{X}}\left\{f(\mathbf{X})+\sum_i\lambda_i(\|\mathbf{a}_i(\mathbf{X})\|^2-|a_i(\mathbf{X})|^2)\right\}. \end{aligned} \quad (88)$$

■

[4]The Lagrangian is formulated by subtracting the product of the dual cones with the primal cones. The products are subtracted instead of added (as in regular convex programming) because the SOC is defined as a "*greater than or equal*" generalized inequality and not as a "*less than or equal*" generalized inequality [21].

Appendix II
Properties of $f_i(\mathbf{\Lambda})$

Consider the functions

$$f_i(\mathbf{\Lambda})=\frac{\gamma_o}{1+\gamma_o}\frac{1}{[\mathbf{H}\mathbf{G}^\dagger(\lambda_i)\mathbf{H}^H]_{i,i}}, \quad i=1,\ldots,K. \quad (89)$$

In this appendix, we will prove some properties of $f_i(\mathbf{\Lambda})$. For simplicity and due to space limitations, we will only deal with real-valued variables and functions. The proofs rely on the following proposition.

Proposition 4: If $\mathbf{A}\succeq 0, \mathbf{B}\succeq 0$ and $\mathbf{c}$ is in the range of $\mathbf{A}$, then

$$\frac{1}{\mathbf{c}^T(\mathbf{A}+\mathbf{B})^\dagger\mathbf{c}}\geq\frac{1}{\mathbf{c}^T\mathbf{A}^\dagger\mathbf{c}} \quad (90)$$

with equality if and only if $\mathbf{B}(\mathbf{A}+\mathbf{B})^\dagger\mathbf{c}=\mathbf{0}$.

Proof: First note that if $\mathbf{c}$ is in the range of $\mathbf{A}\succeq 0$, then

$$\mathcal{Q}(\mathbf{A},\mathbf{c})=\frac{1}{\mathbf{c}^T\mathbf{A}^\dagger\mathbf{c}}=\begin{cases}\min_{\mathbf{x}} & \mathbf{x}^T\mathbf{A}\mathbf{x}\\ \text{s.t.} & \mathbf{c}^T\mathbf{x}=1.\end{cases} \quad (91)$$

As proof, let us derive the Lagrangian of (91)

$$\mathcal{L}=\mathbf{x}^T\mathbf{A}\mathbf{x}+\lambda(\mathbf{c}^T\mathbf{x}-1). \quad (92)$$

Equating the derivative with respect to $\mathbf{x}$ to zero yields

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}}=2\mathbf{A}\mathbf{x}+\lambda\mathbf{c}=\mathbf{0}. \quad (93)$$

Clearly, the solution to this condition is $\mathbf{x}=\mathbf{A}^\dagger\mathbf{c}/\mathbf{c}^T\mathbf{A}^\dagger\mathbf{c}+\mathbf{y}$ and $\lambda=-2/(\mathbf{c}^T\mathbf{A}^\dagger\mathbf{c})$, where $\mathbf{y}$ is any vector in the null space of $\mathbf{A}$.

Using (91), we need to prove that

$$\mathcal{Q}(\mathbf{A}+\mathbf{B},\mathbf{c})\geq\mathcal{Q}(\mathbf{A},\mathbf{c}). \quad (94)$$

Let us denote the optimal argument of $\mathcal{Q}(\mathbf{A},\mathbf{c})$ by $\mathbf{x}_A$ and the optimal argument of $\mathcal{Q}(\mathbf{A}+\mathbf{B},\mathbf{c})$ by $\mathbf{x}_{A+B}$. In order to prove the inequality, assume the contrary, i.e., the optimal value of $\mathcal{Q}(\mathbf{A}+\mathbf{B},\mathbf{c})$ is less than that of $\mathcal{Q}(\mathbf{A},\mathbf{c})$. Then, this is a contradiction to the optimality of $\mathbf{x}_A$, because $\mathbf{x}_{A+B}$ is feasible for $\mathcal{Q}(\mathbf{A},\mathbf{c})$ and results in a smaller objective value.

In order to prove the case of strict inequality, we examine the case when $\mathcal{Q}(\mathbf{A}+\mathbf{B},\mathbf{c})=\mathcal{Q}(\mathbf{A},\mathbf{c})$. Thus, we have $\mathbf{x}_A^T\mathbf{A}\mathbf{x}_A=\mathbf{x}_{A+B}^T\mathbf{A}\mathbf{x}_{A+B}+\mathbf{x}_{A+B}^T\mathbf{B}\mathbf{x}_{A+B}$. However, due to the optimality of $\mathbf{x}_A$, we have $\mathbf{x}_A^T\mathbf{A}\mathbf{x}_A\leq\mathbf{x}_{A+B}^T\mathbf{A}\mathbf{x}_{A+B}$. These two conditions hold together if and only if $\mathbf{x}_{A+B}^T\mathbf{B}\mathbf{x}_{A+B}=0$. Plugging in the optimal $\mathbf{x}_{A+B}$ yields $\mathbf{c}^T(\mathbf{A}+\mathbf{B})^\dagger\mathbf{B}(\mathbf{A}+\mathbf{B})^\dagger\mathbf{c}=0$. Finally, due to semidefiniteness of $\mathbf{B}$, this is possible only if $\mathbf{B}(\mathbf{A}+\mathbf{B})^\dagger\mathbf{c}=\mathbf{0}$. ■

Using Proposition 4 with $\mathbf{c}=\mathbf{H}^T\mathbf{e}_i$ (which is in the range of $\mathbf{H}_{\mathrm{Tx}}^T\mathbf{H}_{\mathrm{Tx}}$), we prove the following properties.

- *Positivity*—If $\lambda_i\geq 0$ for $i=1,\ldots,K$, then $f_j(\mathbf{\Lambda})>0$ for $j=1,\ldots,K$.

Proof. Observe the following chain:

$$\begin{aligned}f_i(\mathbf{\Lambda})&=\frac{\gamma_o}{1+\gamma_o}\frac{1}{[\mathbf{H}\mathbf{G}^\dagger(\lambda_i)\mathbf{H}^T]_{i,i}}\\ &\geq\frac{\gamma_o}{1+\gamma_o}\frac{1}{\left[\mathbf{H}\left(\mathbf{H}_{\mathrm{Tx}}^T\mathbf{H}_{\mathrm{Tx}}\right)^\dagger\mathbf{H}^T\right]_{i,i}}, \; i=1,\ldots,K\end{aligned} \quad (95)$$

where we have used Proposition 4 with $\mathbf{A} = \mathbf{H}_{\mathrm{Tx}}^T\mathbf{H}_{\mathrm{Tx}}$ and $\mathbf{B} = \mathbf{H}^T\mathbf{\Lambda}\mathbf{H}$. Due to the semidefiniteness, the diagonal elements of $\mathbf{H}(\mathbf{H}_{\mathrm{Tx}}^T\mathbf{H}_{\mathrm{Tx}})^\dagger\mathbf{H}^T$ are nonnegative. Therefore, if they are finite, then their inverses are strictly positive, and the property holds.

- *Monotonicity*—If $\lambda_i \geq \lambda_i'$ for $i = 1,\ldots,K$, then $f_j(\mathbf{\Lambda}) \geq f_j(\mathbf{\Lambda}')$ for $j = 1,\ldots,K$.

Proof. Observe the following chain:

$$\begin{aligned} f_i(\mathbf{\Lambda}) &= \frac{\gamma_o}{1+\gamma_o}\frac{1}{[\mathbf{H}\mathbf{G}^\dagger(\lambda_i)\mathbf{H}^T]_{i,i}} \\ &= \frac{\gamma_o}{1+\gamma_o}\frac{1}{[\mathbf{H}(\mathbf{G}(\lambda_i') + \mathbf{H}^T\mathrm{diag}\{\lambda_i - \lambda_i'\}\mathbf{H})^\dagger\mathbf{H}^T]_{i,i}} \\ &\geq \frac{\gamma_o}{1+\gamma_o}\frac{1}{[\mathbf{H}\mathbf{G}^\dagger(\lambda_i')\mathbf{H}^T]_{i,i}} \\ &= f_i(\mathbf{\Lambda}'), \quad i = 1,\ldots,K \end{aligned} \tag{96}$$

where we have used Proposition 4 with $\mathbf{A} = \mathbf{G}(\lambda_i')$ and $\mathbf{B} = \mathbf{H}^T\mathrm{diag}\{\lambda_i - \lambda_i'\}\mathbf{H}$.

- *Scalability*—If $\alpha > 1$, then $\alpha f_i(\mathbf{\Lambda}) > f_i(\alpha\mathbf{\Lambda})$ for $j = 1,\ldots,K$.

Proof. Observe the following chain:

$$\begin{aligned} \alpha f_i(\mathbf{\Lambda}) &= \alpha\frac{\gamma_o}{1+\gamma_o}\frac{1}{[\mathbf{H}\mathbf{G}^\dagger(\lambda_i)\mathbf{H}^T]_{i,i}} \\ &= \frac{\gamma_o}{1+\gamma_o}\frac{1}{\left[\mathbf{H}\left(\mathbf{G}(\alpha\lambda_i) + (\alpha-1)\mathbf{H}_{\mathrm{Tx}}^T\mathbf{H}_{\mathrm{Tx}}\right)^\dagger\mathbf{H}^T\right]_{i,i}} \\ &\geq \frac{\gamma_o}{1+\gamma_o}\frac{1}{[\mathbf{H}\mathbf{G}^\dagger(\alpha\lambda_i)\mathbf{H}^T]_{i,i}} \\ &= f_i(\alpha\mathbf{\Lambda}), \quad i = 1,\ldots,K, \end{aligned} \tag{97}$$

where we have used Proposition 4 with $\mathbf{A} = \mathbf{G}(\alpha\lambda_i)$ and $\mathbf{B} = (\alpha-1)\mathbf{H}_{\mathrm{Tx}}^T\mathbf{H}_{\mathrm{Tx}}$. The inequality is non strict, i.e., holds with equality if and only if $\mathbf{H}_{\mathrm{Tx}}(\mathbf{A}+\mathbf{B})^\dagger\mathbf{H}^T\mathbf{e}_i = \mathbf{0}$. Multiplying by $\mathbf{e}_i^T\mathbf{H}_{\mathrm{RxCh}}$ on the left, yields $\mathbf{e}_i^T\mathbf{H}(\mathbf{A}+\mathbf{B})^\dagger\mathbf{H}^T\mathbf{e}_i = \mathbf{0}$. Therefore, $\mathbf{H}^T\mathbf{e}_i \in \mathcal{N}\{(\mathbf{A}+\mathbf{B})^\dagger\}$, and due to the symmetry of $\mathbf{A}+\mathbf{B}$, we also have $\mathbf{H}^T\mathbf{e}_i \in \mathcal{N}\{\mathbf{A}+\mathbf{B}\}$. In addition, due to the semidefiniteness, this means that $\mathbf{H}^T\mathbf{e}_i \in \mathcal{N}\{\mathbf{H}^T\mathrm{diag}\{\alpha\lambda_i\}\mathbf{H}\}$ and $\mathbf{H}^T\mathbf{e}_i \in \mathcal{N}\{\mathbf{H}_{\mathrm{Tx}}^T\mathbf{H}_{\mathrm{Tx}}\}$. Therefore, $\mathbf{H}_{\mathrm{Tx}}^T\mathbf{H}_{\mathrm{Tx}}\mathbf{H}_{\mathrm{Tx}}^T\mathbf{H}_{\mathrm{ChRx}}^T\mathbf{e}_i = \mathbf{0}$, and $\mathbf{H}_{\mathrm{Tx}}^T\mathbf{H}_{\mathrm{ChRx}}^T\mathbf{e}_i = \mathbf{0}$. Consequently, the problem is infeasible.

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